

Fractional calculus and zeta functions

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Analytic number theory

Analytic number theory is the part of mathematics that brings together analysis with number theory. Some properties and ideas from number theory can be described using analytic functions, e.g. zeta functions.

Often this field is studied from the viewpoint of number theory (e.g. Dirichlet L-functions, which are like zeta functions modulo a prime p). I want to use the viewpoint of analysis.

Riemann, Hurwitz, Lerch zeta functions

The **Riemann zeta function** is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \operatorname{Re}(s) > 1.$$

The **Hurwitz zeta function** is defined by

$$\zeta(x, s) = \sum_{n=0}^{\infty} (n+x)^{-s}, \quad \operatorname{Re}(s) > 1, \operatorname{Re}(x) > 0.$$

The **Lerch zeta function** is defined by

$$L(t, x, s) = \sum_{n=0}^{\infty} (n+x)^{-s} e^{2\pi i t n}, \quad \operatorname{Re}(s) > 1, \operatorname{Re}(x) > 0, \operatorname{Im}(t) \geq 0.$$

All of these extend to all $s \in \mathbb{C}$ by analytic continuation.

Why are these functions interesting?

The Riemann zeta function can be used to describe the distribution of prime numbers within the natural numbers. One of its most important features is the **Euler product formula**:

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \operatorname{Re}(s) > 1.$$

This emphasises the connection with prime numbers.

The zeta function itself is an analytic function of a complex variable. It can be studied analytically, but it turns out to be very hard to prove any properties for it. Many interesting mathematical techniques have been invented just through studying this function.

Relationships between these functions

- Riemann $\subseteq$ Hurwitz $\subseteq$ Lerch:

$$\zeta(s) = \zeta(1, s), \quad \zeta(x, s) = L(0, x, s).$$

- By adding additional parameters x and t , we find some interesting results which do not have analogues for $\zeta(s)$.
- But we also lose some useful properties, e.g. there is no Euler product formula for Hurwitz or Lerch.

Domains

If $\operatorname{Re}(s) > 1$, then $\zeta(s)$ is **easy** (series):

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

If $\operatorname{Re}(s) < 0$, then $\zeta(s)$ is **still easy** (functional equation):

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s).$$

If $0 < \operatorname{Re}(s) < 1$, then $\zeta(s)$ is **very hard**.

This is the so-called **critical strip**, where all the challenges are.

Riemann Hypothesis

Riemann Hypothesis: about the value of σ if $\zeta(\sigma + it) = 0$.

All non-trivial zeroes of the Riemann zeta function have real part $\frac{1}{2}$.

Critical line $\sigma = \frac{1}{2}$: line of symmetry for the zeta function.

- Posed by Riemann in 1859.
- #8 on Hilbert's list of 23 mathematical open problems (1900).
- One of the Clay Millennium Problems (2000).
- \$1 million for a solution.

Lindelöf Hypothesis

Lindelöf Hypothesis: about the growth of $\zeta(\sigma + it)$ as $t \rightarrow \infty$.

The Riemann zeta function on the critical line grows more slowly than any positive power as the line goes to infinity.

Weaker problem (Riemann $\Rightarrow$ Lindelöf).

Let $\mu = \mu(\sigma)$ be s.t. $\zeta(\sigma + it) = \mathcal{O}(t^{\mu+\epsilon})$ as $t \rightarrow \infty$ for all $\epsilon > 0$.

- By the series formula, $\mu(\sigma) = 0$ for all $\sigma > 1$;
- by the functional equation, $\mu(\sigma) = \frac{1}{2} - \sigma$ for all $\sigma < 0$;
- the function μ is continuous, convex, decreasing, non-negative;
- so $\mu(\frac{1}{2}) \leq \frac{1}{4} = \mathbf{0.25}$.

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Lindelöf Hypothesis: $\mu(\frac{1}{2}) = \mathbf{0}$.

Fractional calculus

Fractional calculus is the study of differentiation and integration to arbitrary non-integer orders. How do we define the $(\frac{1}{2})$ th derivative, $(-\pi)$ th derivative, $(2 + i)$ th derivative, of a function?

Many possible answers to this question. Some different formulae which are equivalent, some completely different definitions. Fractional calculus is more complicated than classical calculus!

The idea goes back to Leibniz, almost as old as calculus itself. But in recent decades the research output in fractional calculus has increased massively.

Why study fractional calculus?

From the mathematical point of view, it's a natural generalisation of the concepts of differentiation and integration from classical calculus. But is it more than just a pure mathematical curiosity?

Many applications of fractional calculus to the real world.

- **Viscoelasticity:** materials which are somehow 'between' viscous liquids and elastic solids. This behaviour is best captured by derivatives 'between' two natural number orders.
- **Non-local interactions:** in fractional calculus, derivatives as well as integrals are non-local operators. So fractional DEs are better for modelling non-local behaviour.
- **Chaos theory** is strongly related to fractal geometry, shapes of non-integer dimension. This has a natural link to fractional calculus, derivatives of non-integer order.

Riemann–Liouville fractional calculus

The most common/fundamental definition is **Riemann–Liouville**. Fractional integrals and derivatives here are defined respectively by:

$${}_c I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_c^t (t-u)^{\alpha-1} f(u) \, du, \quad \operatorname{Re}(\alpha) > 0,$$

$${}_c D_t^\alpha f(t) = \frac{d^n}{dt^n} \left({}_c I_t^{n-\alpha} f(t) \right), \quad n := [\operatorname{Re}(\alpha)] + 1, \quad \operatorname{Re}(\alpha) \geq 0.$$

Here c is a constant of integration, often taken as either 0 or $-\infty$. In fractional calculus, c is needed for differentiation as well as integration.

Riemann–Liouville fractional calculus – justification

- Write ${}_c D_t^{-\alpha} f(t) = {}_c I_t^{\alpha} f(t)$ to get a function ${}_c D_t^{\alpha} f(t)$ defined for **all** $\alpha \in \mathbb{C}$. This is an **analytic function of α** , called a fractional **differintegral**. The RL derivative is the natural extension (analytic continuation) of the RL integral.
- Different values of c are useful, so let's keep the general c :

$${}_0 D_t^{\alpha} (t^{\beta}) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha} \quad , \quad {}_{-\infty} D_t^{\alpha} (e^{kt}) = k^{\alpha} e^{kt}.$$

- Fourier and Laplace transforms, Leibniz rule, ...

The composition of RL integrals is an RL integral (semigroup property), but this is *not* true for RL derivatives.

The topic of this talk

Zeta functions are the subject of some of the hardest and most famous open problems in mathematics. Throughout its 150-year history, analytic number theory has attracted some of the greatest minds in mathematics.

Fractional calculus is an old idea which has been mostly neglected until the last 50 years. Recently it's grown massively as a field of study, discovering applications in many fields of science.

Two important parts of analysis. Why are they never united?

Jerry Bruce Keiper (1953–1995)

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Published in The Mathematica Journal (Spring 1995), Vol. 5, issue 2.

Jerry Keiper, leader of the numerics research and development group at Wolfram Research, was killed in a bicycle accident on January 18, 1995 at the age of 41.

Keiper's life was a rare and wonderful mixture of brilliance and achievement with modesty and humanity. He was driven by a profound desire to do good in the world, while not burdening it with any of his own personal needs.

Keiper was born in Medina, Ohio on October 20, 1953, the second of eight children. He spent his early years on the family farm. Then, after graduating from high school, he enrolled in a technical school, planning to become an electronic technician. But he excelled in mathematics, and even though none of his family had ever gone to college before, he decided to enroll at Ohio State University. He received a bachelor's degree in mathematics from there in 1974, and a master's degree a year later. His master's thesis showed that the Riemann zeta function could be expressed as a fractional derivative of the gamma function—the first of many results he was to obtain about special functions.

Since the mid-1970s, Keiper had maintained a keen interest in analytic number theory and its investigation by computer. In early 1988, Keiper used a prototype of *Mathematica* to explore various relations between zeros of the Riemann zeta function. He hesitantly wrote to D. H. Lehmer, a pioneer of computational number theory, describing his results, and Lehmer replied warmly, encouraging him to publish what he had discovered.

In the course of the next several years, Keiper began to overcome his shyness, and to publish some of his mathematical work. He was particularly interested in finding formulations of the Riemann Hypothesis that would make it more amenable to investigation by numerical methods. He did many large computer experiments both on the ordinary Riemann zeta function and on generalizations and related functions such as the Ramanujan tau functions. A few months before he died, Keiper told me he felt he had made considerable progress. And when he died there were several of his programs found on computers at Wolfram Research that had been running for more than 2000 CPU hours—generating results intended for Keiper to interpret.

Although Keiper did his work on the zeta function mainly to investigate basic questions in number theory, he always made sure that relevant pieces were integrated into *Mathematica*. And in 1990 it was his work that made possible the six-foot-long poster of the Riemann zeta function that Wolfram Research produced for the International Congress of Mathematicians in Kyoto. This poster is now to be found displayed in most mathematics departments around the world. (A special new memorial edition of the poster is being produced.)

Keiper's master's thesis

Jerry Bruce Keiper, Fractional calculus and its relationship to Riemann's zeta function, MSc thesis, Ohio State University, 1975 (37 pages).

Fractional calculus and zeta functions combined for the first time!

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Some New Fractional-Calculus Connections between Mittag-Leffler Functions

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Keiper's methodology (1)

Start from the Hurwitz zeta function and apply a differintegral w.r.t. x :

$${}_{-\infty}D_x^\alpha \zeta(x, s) = \sum_{n=0}^{\infty} {}_{-\infty}D_x^\alpha (n+x)^{-s} = \sum_{n=0}^{\infty} (-1)^{-\alpha} \frac{\Gamma(s+\alpha)}{\Gamma(s)} (n+x)^{-s-\alpha}.$$

Assuming $\operatorname{Re}(\alpha) > 0$ as well as $\operatorname{Re}(s) > 1$, we get:

$$\begin{aligned} {}_{-\infty}D_x^\alpha \left[\zeta(s) - \zeta(x, s) \right] &= (-1)^{-\alpha-1} \frac{\Gamma(s+\alpha)}{\Gamma(s)} \sum_{n=0}^{\infty} (n+x)^{-s-\alpha} \\ &= (-1)^{-\alpha-1} \frac{\Gamma(s+\alpha)}{\Gamma(s)} \zeta(x, s+\alpha). \end{aligned} \quad (1)$$

Keiper's methodology (2)

It is known that (where $\psi = \frac{\Gamma'}{\Gamma}$ is the digamma function and γ is the Euler-Mascheroni constant):

$$\lim_{s \rightarrow 1} [\zeta(s) - \zeta(x, s)] = \psi(x) + \gamma$$

Therefore, letting $s \rightarrow 1$ in (1) gives:

$$-_{\infty} D_x^{\alpha} [\psi(x) + \gamma] = (-1)^{-\alpha-1} \Gamma(1 + \alpha) \zeta(x, 1 + \alpha).$$

Noting that $\operatorname{Re}(\alpha) > 0$, we have:

$$\zeta(x, 1 + \alpha) = \frac{(-1)^{\alpha+1}}{\Gamma(1 + \alpha)} -_{\infty} D_x^{\alpha} \psi(x).$$

So the **Hurwitz zeta function is a fractional derivative of the digamma function**. Note that the variable s in $\zeta(x, s)$ appears as the order of differentiation, and we require $\operatorname{Re}(s) > 1$.

Keiper's main result

$$\zeta(x, s) = \frac{(-1)^s}{\Gamma(s)} {}_{-\infty}D_x^{s-1} \psi(x), \quad \operatorname{Re}(s) > 1.$$

$$\zeta(s) = \frac{(-1)^s}{\Gamma(s)} {}_{-\infty}D_x^{s-1} \psi(x) \Big|_{x=1}, \quad \operatorname{Re}(s) > 1.$$

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THEOREM XIII For $\operatorname{Re}(s) > 1$.

$$\zeta(s) = \sum_{k=1}^{m-1} k^{-s} + \frac{(-1)^s}{\Gamma(s)} \lim_{|c| \rightarrow \infty} {}_cD_z^{s-1} \psi(z) \Big|_{z=m} \bullet$$

Other fractional–zeta connections

Keiper's work is largely **unrecognised**. But two other groups of researchers have worked, in some way, on connections between fractional calculus and zeta functions.

- **Emanuel Guariglia** et al [5, 6, 7] have been working on fractional derivatives of the Riemann zeta function, using a model of fractional calculus which is not the standard Riemann–Liouville one.
- **Hari Srivastava** et al [8, 9, 10] have been working on fractional-calculus relationships between modified Hurwitz–Lerch zeta functions with several extra parameters.

My recent paper

A.F., “The Lerch zeta function as a fractional derivative”, Banach Center Publications 118 (2019), pp. 113–124, DOI 10.4064/bc118-7. Preprint arXiv:1804.07936.

In this paper, inspired by Keiper’s work, I discovered a new way of writing zeta functions as fractional derivatives. This time the starting point is the Lerch zeta function $L(t, x, s)$, and the extra parameter t is vital in the proof, but an interesting result also emerges for the Riemann zeta function $\zeta(s)$.

The key idea

$$\begin{aligned}
 L(t, x, s) &= \sum_{n=0}^{\infty} (n+x)^{-s} e^{2\pi i t n} \\
 &= (2\pi i)^s e^{-2\pi i t x} \sum_{n=0}^{\infty} (2\pi i)^{-s} (n+x)^{-s} e^{2\pi i t (n+x)} \\
 &= (2\pi i)^s e^{-2\pi i t x} \sum_{n=0}^{\infty} {}_{-\infty}D_t^{-s} \left(e^{2\pi i t (n+x)} \right) \\
 &= (2\pi i)^s e^{-2\pi i t x} {}_{-\infty}D_t^{-s} \left(e^{2\pi i t x} \sum_{n=0}^{\infty} (e^{2\pi i t})^n \right) \\
 &= (2\pi i)^s e^{-2\pi i t x} {}_{-\infty}D_t^{-s} \left(\frac{e^{2\pi i t x}}{1 - e^{2\pi i t}} \right).
 \end{aligned}$$

Extending the domain of validity

The proof above used the series formula for zeta, so needs

$$\operatorname{Re}(s) > 1, \quad \operatorname{Re}(x) > 0, \quad \operatorname{Im}(t) > 0.$$

By using analytic continuation and functional equation properties of the Lerch zeta function, we can relax these assumptions to

$$\operatorname{Im}(t) > 0, \quad \operatorname{Im}(x) \geq 0, \quad x \notin (-\infty, 0].$$

Theorem

$$L(t, x, s) = (2\pi)^s \exp \left[i\pi \left(\frac{s}{2} - 2tx \right) \right] {}_{-\infty}D_t^{-s} \left(\frac{e^{2\pi itx}}{1 - e^{2\pi it}} \right) \quad (2)$$

for any $s, x, t \in \mathbb{C}$ with $\operatorname{Im}(t) > 0$ and $\operatorname{Im}(x) \geq 0, x \notin (-\infty, 0]$.

The result for the Riemann zeta function

The Riemann zeta function can be written as

$$\zeta(s) = \frac{(2\pi i)^s}{2^{1-s} - 1} {}_{-\infty}D_t^{-s} \left(\frac{1}{e^{-2\pi i t} - 1} \right) \Bigg|_{t=\frac{1}{2}}$$

for any $s \in \mathbb{C} \setminus \{1\}$, or alternatively as

$$\zeta(s) = (2\pi i)^s \lim_{t \rightarrow 0} \left({}_{-\infty}D_t^{-s} \left(\frac{1}{e^{-2\pi i t} - 1} \right) \right)$$

under the assumption $\operatorname{Re}(s) > 1$.

Continuation of the above results

A.F., Jean-Daniel Djida, “Fractional differential relations for the Lerch zeta function”, under review (2019). Preprint arXiv:2006.01046.

In this paper, we started from the Theorem above on the Lerch zeta function as a fractional derivative, and applied various derivatives to it in order to obtain new fractional results and relations concerning the Lerch zeta function.

Partial t -derivatives

If we know $L(t, x, s) = (2\pi)^s \exp \left[i\pi \left(\frac{s}{2} - 2tx \right) \right] {}_{-\infty}D_t^{-s} \left(\frac{e^{2\pi itx}}{1 - e^{2\pi it}} \right)$,
what can we say about the t -derivatives of this function?

$$\begin{aligned}
 {}_{-\infty}D_t^\alpha L(t, x, s) &= (2\pi i)^s {}_{-\infty}D_t^\alpha \left[e^{-2i\pi tx} {}_{-\infty}D_t^{-s} \left(\frac{e^{2\pi itx}}{1 - e^{2\pi it}} \right) \right] \\
 &= (2\pi i)^s \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{d^n}{dt^n} \left(e^{-2\pi itx} \right) {}_{-\infty}D_t^{\alpha-n} \left[{}_{-\infty}D_t^{-s} \left(\frac{e^{2\pi itx}}{1 - e^{2\pi it}} \right) \right] \\
 &= (2\pi i)^s \sum_{n=0}^{\infty} \binom{\alpha}{n} (-2\pi ix)^n e^{-2\pi itx} {}_{-\infty}D_t^{-s+\alpha-n} \left(\frac{e^{2\pi itx}}{1 - e^{2\pi it}} \right) \\
 &= (2\pi i)^\alpha \sum_{n=0}^{\infty} \binom{\alpha}{n} (-x)^n (2\pi i)^{s-\alpha+n} e^{-2\pi itx} {}_{-\infty}D_t^{-s+\alpha-n} \left(\frac{e^{2\pi itx}}{1 - e^{2\pi it}} \right),
 \end{aligned}$$

using the fractional Leibniz rule and assuming $\operatorname{Re}(s) > 0$.

Partial t -derivatives: result

Theorem

$$-_{\infty}D_t^{\alpha}L(t, x, s) = (2\pi i)^{\alpha} \sum_{n=0}^{\infty} \binom{\alpha}{n} (-x)^n L(t, x, s - \alpha + n),$$

for any $s, x, t, \alpha \in \mathbb{C}$ with $\text{Im}(t) > 0$, $\text{Im}(x) \geq 0$, $x \notin (-\infty, 0]$, $\text{Re}(s) > 0$.

Partial x -derivatives

If we know $L(t, x, s) = (2\pi)^s \exp \left[i\pi \left(\frac{s}{2} - 2tx \right) \right] {}_{-\infty}D_t^{-s} \left(\frac{e^{2\pi itx}}{1 - e^{2\pi it}} \right)$,
what can we say about the x -derivatives of this function?

$$\begin{aligned}
 \frac{\partial}{\partial x} L(t, x, s) &= (2\pi i)^s \frac{\partial}{\partial x} \left[e^{-2i\pi tx} {}_{-\infty}D_t^{-s} \left(\frac{e^{2\pi itx}}{1 - e^{2\pi it}} \right) \right] \\
 &= (2\pi i)^s \left[(-2i\pi t) e^{-2i\pi tx} {}_{-\infty}D_t^{-s} \left(\frac{e^{2\pi itx}}{1 - e^{2\pi it}} \right) + e^{-2i\pi tx} {}_{-\infty}D_t^{-s} \left(\frac{2\pi it e^{2\pi itx}}{1 - e^{2\pi it}} \right) \right] \\
 &= (2\pi i)^s e^{-2i\pi tx} \left[(-2i\pi t) {}_{-\infty}D_t^{-s} \left(\frac{e^{2\pi itx}}{1 - e^{2\pi it}} \right) \right. \\
 &\quad \left. + 2\pi i \left(t {}_{-\infty}D_t^{-s} \left(\frac{e^{2\pi itx}}{1 - e^{2\pi it}} \right) - s {}_{-\infty}D_t^{-s-1} \left(\frac{e^{2\pi itx}}{1 - e^{2\pi it}} \right) \right) \right] \\
 &= (-s)(2\pi i)^{s+1} e^{-2i\pi tx} {}_{-\infty}D_t^{-s-1} \left(\frac{e^{2\pi itx}}{1 - e^{2\pi it}} \right),
 \end{aligned}$$

using the fact that ${}_aD_z^\nu (zf(z)) = z[{}_aD_z^\nu f(z)] + \nu[{}_aD_z^{\nu-1} f(z)]$.

Partial x -derivatives: result

The above result for 1st-order derivatives can be extended by induction:

$$\begin{aligned}\frac{\partial^k}{\partial x^k} L(t, x, s) &= \frac{\partial^{k-1}}{\partial x^{k-1}} [(-s)L(t, x, s+1)] = \frac{\partial^{k-2}}{\partial x^{k-2}} [(-s)(-s-1)L(t, x, s+2)] \\ &= \cdots = (-s)(-s-1)(-s-2) \cdots (-s-k+1)L(t, x, s+k) \\ &= \frac{\Gamma(1-s)}{\Gamma(1-s-k)} L(t, x, s+k).\end{aligned}$$

Theorem

$$\frac{\partial^k}{\partial x^k} L(t, x, s) = \frac{\Gamma(1-s)}{\Gamma(1-s-k)} L(t, x, s+k),$$

for s, x, t as before and $k \in \mathbb{N}$.

It can also be extended to fractional x -derivatives.

Comparison of results

- **Keiper:** $\zeta(x, s)$ as s th fractional derivative of $\psi(x)$.
Complexity of zeta encoded by fractional calculus.
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- **Guariglia:** behaviour of fractional derivatives of $\zeta(s)$.
Interesting study, but no new expression for zeta itself.
- **Srivastava:** fractional links between multi-parameter zeta functions.
No new information about the original $\zeta(s)$.

Comparison of results

- **Keiper:** $\zeta(x, s)$ as s th fractional derivative of $\psi(x)$.
Complexity of zeta encoded by fractional calculus.
Only valid for $\text{Re}(s) > 1$ (easy region for zeta).
- **Guariglia:** behaviour of fractional derivatives of $\zeta(s)$.
Interesting study, but no new expression for zeta itself.
- **Srivastava:** fractional links between multi-parameter zeta functions.
No new information about the original $\zeta(s)$.
- **F.:** $L(t, x, s)$ as s th fractional integral of elementary function.
All complexity of zeta encoded by fractional calculus.
Valid for all s in complex plane.

Conclusions

The Riemann (Hurwitz, Lerch) zeta functions can be written as a **fractional differintegral of an elementary function**.

- All the complexity of the zeta function (very complex!) can be encoded by a single fractional differintegration operation.
- The variable s is the order of differintegration. So finding zeta zeros equates to solving a “differential equation” like

$${}_cD^x f(a) = 0.$$

- Fractional calculus can be used to attack zeta functions. Perhaps a better understanding of fractional calculus may give new approaches to the Riemann and Lindelöf Hypotheses.

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Thank You!