Nonlocal fractional Monge–Ampère equations

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The fractional Laplacian

Fractional powers of matrices

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This exponential solves a "heat equation":

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\begin{cases}
\partial_t(e^{-tA}) = -A(e^{-tA}) & \text{for all } t > 0 \\
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It is easy to verify that for any $\lambda > 0$ and $0 < s < 1$

$$\lambda^s = c_s \int_0^\infty \left(e^{-t\lambda} - 1\right) \frac{dt}{t^{1+s}}$$

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Thus,

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&= c_s P^T \operatorname{diag}\left[\int_0^\infty (e^{-t\lambda_1} - 1) \frac{dt}{t^{1+s}}, \dots, \int_0^\infty (e^{-t\lambda_n} - 1) \frac{dt}{t^{1+s}}\right] P \\
&= c_s \int_0^\infty \left[P^T \operatorname{diag}(e^{-t\lambda_1} - 1, \dots, e^{-t\lambda_n} - 1) P\right] \frac{dt}{t^{1+s}} \\
&= c_s \int_0^\infty \left[P^T \operatorname{diag}(e^{-t\lambda_1}, \dots, e^{-t\lambda_n}) P - P^T (Id) P\right] \frac{dt}{t^{1+s}} \\
&= c_s \int_0^\infty \left[e^{-tA} - Id\right] \frac{dt}{t^{1+s}}
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- Using e^{-tA} we can also compute the matrices A^{-s}, $s > 0$.

Fourier transform (1822)

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$$\mathcal{F}(u)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-i\xi\cdot x}\, dx$$

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- **Property 2.** Derivatives correspond to multiplication by polynomials:

$$D^k u \quad \longleftrightarrow \quad (i\xi)^k \mathcal{F}(u)$$

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Then $e^{t\Delta}u(x)$ solves the heat equation

$$\begin{cases} \partial_t(e^{t\Delta}u) = \Delta(e^{t\Delta}u) & \text{for all } t > 0 \\ e^{t\Delta}u\big|_{t=0} = u \end{cases}$$

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Moreover, the exponential is given by convolution with the *heat kernel*

$$e^{t\Delta}u(x) = \int_{\mathbb{R}^n} \frac{e^{-|x-y|^2/(4t)}}{(4\pi t)^{n/2}}\, u(y)\, dy$$

Fractional Laplacian and exponential

In this case we use: $$|\xi|^{2s} = c_s \int_0^\infty \left(e^{-t|\xi|^2} - 1\right) \frac{dt}{t^{1+s}} \quad (0 < s < 1)$$

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- Nonlocal - Fractional order $0 < 2s < 2$

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- See S., User's guide to the fractional Laplacian and the method of semigroups, *Handbook of Fractional Calculus with Applications*, Vol. 2, De Gruyter, 2019.

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If we divide by h^2 and take the limit as $h \to 0$,

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so that u is a harmonic function.

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We also proved approximation of continuous problems $(-\Delta)^s u = f$ by discrete problems $(-\Delta_h)^s u_h = f_h$, under **minimal regularity** of f, and with **explicit rates** of convergence in the L^∞-**norm**.

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- Image processing: digital images are discontinuous functions.

Examples of kernels

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The Monge–Ampère equation

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- Elliptic MA equation and convexity go together.

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The MA equation is a concave function of $D^2 u$.

Claim. If u is convex and C^2 then

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\begin{aligned}
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Proof. Arithmetic mean-geometric mean inequality:

$$[\det(D^2u)]^{1/n} = (\lambda_1\cdots\lambda_n)^{1/n} \le \frac{\lambda_1 + \cdots + \lambda_n}{n} = \frac{1}{n}\operatorname{trace}(Id^2\cdot D^2u)$$

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MA is degenerate elliptic. In dimension $n = 2$, matrices of the form

$$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1/\varepsilon \end{pmatrix}$$

enter in the computation of the infimum.

The Monge–Ampère equation and uniform ellipticity

Recall: if u is convex and C^2 then

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Conclusion. The matrices A have minimum eigenvalue uniformly bounded below away from zero and, therefore, above away from infinity: **uniform ellipticity**.

Nonlocal fractional Monge–Ampère equations

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By using the integral formula for the fractional Laplacian,

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A fractional Monge–Ampère equation

Recall: if u is convex and C^2 then

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- This fractional MA operator is **degenerate elliptic**.

Fractional MA equation

Caffarelli and Charro considered the Dirichlet problem

$$\begin{cases} \mathcal{D}_s \bar{u} = \bar{u} - \phi & \text{in } \mathbb{R}^n \\ \lim_{|x|\to\infty} (\bar{u} - \phi)(x) = 0 \end{cases}$$

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- The uniformly elliptic regularity theory of Caffarelli–Silvestre applies.

Obstacle problem for fractional MA equation

We consider the obstacle problem

$$\begin{cases} \mathcal{D}_s u \geq u - \phi & \text{in } \mathbb{R}^n \\ u \leq \psi & \text{in } \mathbb{R}^n \\ \mathcal{D}_s u = u - \phi & \text{in } \{u < \psi\} \\ \lim_{|x|\to\infty}(u-\phi)(x) = 0 \end{cases}$$

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Theorem (with Y. Jhaveri, *Comm. PDE* 2020)

There exists a unique viscosity solution u that is Lipschitz and semiconcave with constants depending only on ϕ and ψ. Moreover

$$u > \phi \quad \textit{in } \mathbb{R}^n$$

In particular, locally, the problem becomes uniformly elliptic.
Higher regularity of u and regularity of the free boundary $\partial\{u < \psi\}$.

Nonlocal uniformly elliptic equations

We have

$$\begin{aligned}
\mathcal{D}_s^\lambda u(x) &= \inf_{A>\lambda Id,\det(A)=1} c \int_{\mathbb{R}^n} \frac{u(x-y)-u(x)}{|A^{-1}y|^{n+2s}}\,dy \\
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Theorem (with M. Vaughan, *PhD thesis* 2020, arXiv preprint)

Suppose that $B(x)$ is uniformly elliptic and Hölder continuous. If $u \geq 0$ is a classical solution to $(-L)^s u = f$ in B_1 then

$$\sup_{B_{1/2}} u \leq C\Big(\inf_{B_{1/2}} u + \|f\|_{L^\infty(B_1)}\Big).$$

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- Under the semigroup setting given by Galé–Miana–S. in *J. Evol. Equ.* 2012, we developed a new method of sliding paraboloids for singular/degenerate PDEs.

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Recall that $v = \partial_e u$ solves the linearized MA equation

$$Lv = \mathrm{trace}(\det(D^2u)(D^2u)^{-1}D^2v) = \mathrm{trace}(A_u(x)D^2v) = g$$

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- Under the semigroup setting given by Galé–Miana–S. in *J. Evol. Equ.* 2012, we exploit the divergence and nondivergence structures of the equation.

Thank you for your attention!