# Controllability of space-time fractional diffusive and super diffusive equations

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# Outline

## The considered problem

In this talk we consider the following system of evolution equation:

$$\begin{cases} \partial_t^\alpha u(t,x) + (-\Delta)^s u(t,x) = f\chi_\omega \quad \text{in} \;\; \Omega \times (0,T), \\ + \text{ Intial conditions}, \\ + \text{ Boundary conditions}. \end{cases} \tag{1.1}$$

Here $\alpha > 0$ is a real number, $0 < s \leq 1$, $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz continuous boundary $\partial\Omega$, $(-\Delta)^s$ is the fractional Laplacian and $\partial_t^\alpha$ is a fractional time derivative of Caputo type. These notions will be defined later.

- If $\alpha = 1$ (resp. $\alpha = 2$) we have the heat (resp. wave) equation.
- If $0 < \alpha < 1$ such an equation is said to be of slow diffusion.
- If $1 < \alpha < 2$, then it is said to be of super diffusion.

## Questions

- How to define the fractional Laplace operator $(-\Delta)^s$?
- How to define a time fractional derivative $\partial_t^\alpha$?
- Which initial and boundary conditions make the system (1.1) well posed as a Cauchy problem?
- Is there a control function $f$ localized in a nonempty open set $\omega \subset \Omega$ such that solutions of the system can rest at some time $T > 0$? In other words, is such system null controllable?
- Given a target, is there a function $f$ localized in a nonempty open set $\omega \subset \Omega$ such that solutions of the system reach the given target at time $T > 0$? In other words, is such system exactly controllable?

# Outline

## The fractional Laplacian: Using Fourier Analysis

Using Fourier analysis, we have that the fractional Laplace operator $(-\Delta)^s$ can be defined as the pseudo-differential operator with symbol $|\xi|^{2s}$. That is,

$$(-\Delta)^s u = C_{N,s}\mathcal{F}^{-1}\left(|\xi|^{2s}\mathcal{F}(u)\right),$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier, and inverse Fourier, transform, respectively, and $C(N,s)$ is an appropriate normalizing constant depending only on $N$ and $s$.

## The fractional Laplacian: Using Singular Integrals

Let $0 < s < 1$ and $\varepsilon > 0$ be real numbers. For a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ we let

$$(-\Delta)^s_\varepsilon u(x) = C_{N,s} \int_{\{y \in \mathbb{R}^N:\ |x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N.$$

The fractional Laplacian $(-\Delta)^s$ is defined for $x \in \mathbb{R}^N$ by the following singular integral:

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy = \lim_{\varepsilon \downarrow 0} (-\Delta)^s_\varepsilon u(x),$$

provided that the limit exists, where $C_{N,s} := \dfrac{s 2^{2s} \Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}$. Here, $\Gamma$ denotes the usual Euler-Gamma function.

## The fractional Laplacian: Using the Caffarelli-Silvestre extension (CPDE, 2007)

Let $0 < s < 1$. For $u : \mathbb{R}^N \to \mathbb{R}$ in an appropriate space, consider the harmonic extension $W : [0, \infty) \times \mathbb{R}^N \to \mathbb{R}$. That is the unique weak solution of the Dirichlet problem

$$\begin{cases} W_{tt} + \frac{1-2s}{t} W_t + \Delta_x W = 0 & \text{in } (0,\infty) \times \mathbb{R}^N, \\ W(0,\cdot) = u & \text{in } \mathbb{R}^N. \end{cases} \tag{2.1}$$

Then the fractional Laplace operator can be defined as

$$(-\Delta)^s u(x) = -d_s \lim_{t \to 0^+} t^{1-2s} W_t(t,x), \quad x \in \mathbb{R}^N,$$

where the constant $d_s$ is given by $d_s := 2^{2s-1} \dfrac{\Gamma(s)}{\Gamma(1-s)}$. This is known in the literature as the Caffarelli-Silvestre extension.

## All the definitions coincide

- Let $0 < s < 1$. Then

$$\begin{aligned}(-\Delta)^s u(x) =& C_{N,s}\mathcal{F}^{-1}\left(|\xi|^{2s}\mathcal{F}(u)\right)\\ =& C_{N,s}\text{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy\\ =& -d_s\lim_{t\to 0+}t^{1-2s}W_t(t,x),\end{aligned}$$

  where we recall that $W:[0,\infty)\times\mathbb{R}^N\to\mathbb{R}$ is the harmonic extension which is the unique weak solution of the Dirichlet problem (2.1).

- It is clear that $(-\Delta)^s$ is a nonlocal operator. That is,

$$\text{supp}[(-\Delta)^s u] \not\subset \text{supp}[u].$$

## Derivation of singular integrals: Long jump random walks

Let $\mathcal{K}: \mathbb{R}^N \to [0,\infty)$ be an even function such that

$$\sum_{k\in\mathbb{Z}^N} \mathcal{K}(k) = 1. \tag{2.2}$$

Given a small $h > 0$, we consider a random walk on the lattice $h\mathbb{Z}^N$.

- We suppose that at any unit time $\tau$ (which may depend on $h$) a particle jumps from any point of $h\mathbb{Z}^N$ to any other point.
- The probability for which a particle jumps from a point $hk \in h\mathbb{Z}^N$ to the point $h\tilde{k}$ is taken to be $\mathcal{K}(k-\tilde{k}) = \mathcal{K}(\tilde{k}-k)$. Note that, differently from the standard random walk, in this process the particle may experience arbitrarily long jumps with small probability.

## Long jump random walks: Continue

- Let $u(x,t)$ be the probability that our particle lies at $x \in h\mathbb{Z}^N$ at time $t \in \tau\mathbb{Z}$.
- Then $u(x, t+\tau)$ is the sum of all the probabilities of the possible positions $x + hk$ at time $t$ weighted by the probability of jumping from $x + hk$ to $x$. That is,

$$u(x, t+\tau) = \sum_{k \in \mathbb{Z}^N} \mathcal{K}(k) u(x + hk, t).$$

- Using (2.2) we get the following evolution law:

$$u(x, t+\tau) - u\,(x,t) = \sum_{k \in \mathbb{Z}^N} \mathcal{K}(k) \left[ u(x+hk, t) - u(x,t) \right]. \quad (2.3)$$

## Long jump random walks: Continue

- In particular, in the case when $\tau = h^{2s}$ and $\mathcal{K}$ is homogeneous i.e.,

$$\mathcal{K}(y) = |y|^{-(N+2s)} \text{ for } \ y \neq 0, \ \mathcal{K}(0) = 0, \ \text{and } 0 < s < 1,$$

  then (2.2) holds and $\mathcal{K}(k)/\tau = h^N \mathcal{K}(hk)$.

- Therefore, we can rewrite (2.3) as follows:

$$\frac{u(x,t+\tau) - u(x,t)}{\tau} = h^N \sum_{k \in \mathbb{Z}^N} \mathcal{K}(hk)\left[u(x+hk,t) - u(x,t)\right]. \tag{2.4}$$

- Notice that the term on the right-hand side of (2.4) is just the approximating Riemann sum of

$$\int_{\mathbb{R}^N} \mathcal{K}(y)\left[u(x+y,t) - u(x,t)\right] dy.$$

## Long jump random walks: Continue

- Thus letting $\tau = h^{2s} \to 0^+$ in (2.4), we obtain the evolution equation

$$\partial_t u(x,t) = \int_{\mathbb{R}^N} \frac{u(x+y,t) - u(x,t)}{|y|^{N+2s}} dy. \tag{2.5}$$

- Notice that (2.5) has a singularity at $y = 0$. But when $0 < s < 1$ and the function $u$ is smooth, then it can be viewed as a Principal Value as we have clarified above. More precisely, we have the following:

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N \setminus B(0,\varepsilon)} \frac{u(x+y,t) - u(x,t)}{|y|^{N+2s}} dy \\
&= \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} \frac{u(z,t) - u(x,t)}{|z-x|^{N+2s}} dz \\
&= -\left(C_{N,s}\right)^{-1} (-\Delta)^s u(x,t).
\end{aligned}$$

## Long jump random walks: Conclusion

We have shown above that a simple random walk with possibly long jumps produces, at the limit, a singular integral with a homogeneous kernel.

## The limit as $s \uparrow 1$

Let $u, v$ be smooth functions with compact support in $\Omega$. That is, $u, v \in \mathcal{D}(\Omega)$. Then the following holds.

$$\lim_{s\uparrow 1^-} \int_{\mathbb{R}^N} v(-\Delta)^s u dx = -\int_\Omega v\Delta u dx = \int_\Omega \nabla u \cdot \nabla v \, dx.$$

## Proof

Using a result due to Bourgain, Brezis and Mironescu we get:

$$\begin{aligned}
&\lim_{s\uparrow 1^-} \int_{\mathbb{R}^N} u(-\Delta)^s u dx \\
=&\lim_{s\uparrow 1} \frac{s 2^{2s-1}\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}}(1-s)\Gamma(1-s)}(1-s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \\
=&\int_{\mathbb{R}^N} |\nabla u|^2 dx = \int_\Omega |\nabla u|^2 dx = -\int_\Omega u\Delta u dx.
\end{aligned}$$

## Question

1. What are the Dirichlet and Neumann Boundary Conditions for the fractional Laplace operator $(-\Delta)^s$?
2. To obtain an explicit and a rigorous answer to the above question, we need the following notions.
    - We need some appropriate Sobolev spaces.
    - We need a notion of a (fractional) normal derivative.
    - We also need an integration by parts formula for $(-\Delta)^s$. That is, an appropriate Green type formula for $(-\Delta)^s$.

## Fractional order Sobolev Spaces

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set and $0 < s < 1$.

- We denote

$$W^{s,2}(\Omega) := \left\{ u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy < \infty \right\},$$

and we endow it with the norm defined by

$$\|u\|_{W^{s,2}(\Omega)} = \left( \int_\Omega |u|^2 \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.$$

- Then $W^{s,2}(\Omega)$ is a Hilbert space.

## Fractional order Sobolev Spaces: Continue

Let $\mathcal{D}(\Omega)$ be the space of test functions on $\Omega$. Let

$$W_0^{s,2}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{s,2}(\Omega)},$$

and

$$W_0^{s,2}(\overline{\Omega}) = \Big\{ u \in W^{s,2}(\mathbb{R}^N) : \ u = 0 \ \text{ a.e. on } \ \mathbb{R}^N \setminus \Omega \Big\}.$$

1. There is no obvious inclusion between $W_0^{s,2}(\Omega)$ and $W_0^{s,2}(\overline{\Omega})$.
2. If $\Omega \subset \mathbb{R}^N$ is Lipschitz, then we have the following situation.
   - If $s \neq \frac{1}{2}$, then $W_0^{s,2}(\Omega) = W_0^{s,2}(\overline{\Omega})$.
   - If $s = \frac{1}{2}$, then $W_0^{s,2}(\overline{\Omega})$ is a proper subspace of $W_0^{s,2}(\Omega)$.
   - For arbitrary bounded open sets, the relation can be found in W. Potential Analysis, 2015.

## The Dirichlet problem for $(-\Delta)^s$

- Let $g \in C(\partial\Omega)$. The classical Dirichlet problem for $\Delta$ is given by

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u = g \text{ on } \quad \partial\Omega.$$

- Let $g \in C(\partial\Omega)$. Then the Dirichlet problem

$$(-\Delta)^s u = 0 \quad \text{in} \quad \Omega, \quad u = g \text{ on } \quad \partial\Omega, \tag{2.6}$$

is not well-posed. This follows from the fact that

$$(-\Delta)^s u(x) = C_{N,s} \int_\Omega \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy + C_{N,s} \int_{\mathbb{R}^N \setminus \Omega} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy.$$

- Let $g \in C_0(\mathbb{R}^N \setminus \Omega)$. The well-posed Dirichlet problem is given by

$$(-\Delta)^s u = 0 \quad \text{in} \quad \Omega, \quad u = g \quad \text{in} \quad \mathbb{R}^N \setminus \Omega.$$

## The zero Dirichlet exterior condition (EC) for $(-\Delta)^s$

1. The zero Dirichlet BC for $\Delta$ is given by $u = 0$ on $\partial\Omega$.
2. The zero Dirichlet EC is characterized by $u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega$.

## How to define a "fractional" normal derivative?

- Recall that if $u$ is a smooth function defined on a smooth open set $\Omega$, then the normal derivative of $u$ is given by

$$\frac{\partial u}{\partial \nu} := \nabla u \cdot \vec{\nu},$$

where $\vec{\nu}$ is the normal vector at the boundary $\partial\Omega$.

- For $0 < s < 1$ and a function $u$ defined on $\mathbb{R}^N$ we let

$$\mathcal{N}_s u(x) = C_{N,s} \int_\Omega \frac{u(x) - u(y)}{|x - y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega},$$

provided that the integral exists. This is clearly a nonlocal operator.

- $\mathcal{N}_s$ is well-defined and continuous from $W^{s,2}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N \setminus \Omega)$.
- We call $\mathcal{N}_s u$ the nonlocal normal derivative of $u$.

## Why is $\mathcal{N}_s$ a normal derivative?

- Recall the divergence theorem:

$$-\int_\Omega \Delta u \, dx = -\int_\Omega \operatorname{div}(\nabla u) \, dx = -\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, d\sigma, \ \ \forall \, u \in C^2(\overline{\Omega}).$$

- For $(-\Delta)^s$ we have the following:

$$\int_\Omega (-\Delta)^s u \, dx = -\int_{\mathbb{R}^N \setminus \Omega} \mathcal{N}_s u \, dx, \ \ \forall \, u \in C_0^2(\mathbb{R}^N).$$

## Why is $\mathcal{N}_s$ a normal derivative?

- Green Formula: $\forall\ u \in C^2(\overline{\Omega})$ and $\forall\ v \in C^1(\overline{\Omega})$,

$$\int_\Omega \nabla u \cdot \nabla v\, dx = -\int_\Omega v \Delta u\, dx + \int_{\partial\Omega} v \frac{\partial u}{\partial \nu}\, d\sigma.$$

- For $(-\Delta)^s$ we have the following: $\forall\ u \in C_0^2(\mathbb{R}^N)$ and $v \in C_0^1(\mathbb{R}^N)$,

$$\frac{C_{N,s}}{2} \int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy$$
$$= \int_\Omega v(-\Delta)^s u\, dx + \int_{\mathbb{R}^N \setminus \Omega} v \mathcal{N}_s u\, dx.$$

$$\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2 = (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega).$$

## Why is $\mathcal{N}_s$ a normal derivative?

For every $u, v \in C_0^2(\mathbb{R}^N)$ we have that (Ros-Oton and Valdinoci, JEMS 2017)

$$\lim_{s\uparrow 1^-} \int_{\mathbb{R}^N\setminus\Omega} v\mathcal{N}_s u\, dx = \int_{\partial\Omega} v\frac{\partial u}{\partial \nu}\, d\sigma.$$

## Observation

We have shown that $\mathcal{N}_s$ plays the same role for $(-\Delta)^s$ that the classical normal derivative $\frac{\partial}{\partial\nu}$ does for $\Delta$.

## The Neumann problem for $(-\Delta)^s$

1. $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$. The Neumann problem for $\Delta$ is given by

$$-\Delta u = f \ \text{ in } \ \Omega, \ \ \frac{\partial u}{\partial \nu} = g \ \text{ on } \ \partial\Omega.$$

It is well-known that the above problem is well-posed if and only if

$$\int_\Omega f\, dx + \int_{\partial\Omega} g\, d\sigma = 0.$$

2. Let $f \in L^2(\Omega)$ and $g \in L^1(\mathbb{R}^N \setminus \Omega)$. We consider the problem

$$(-\Delta)^s u = f \quad \text{in } \ \Omega, \ \ \mathcal{N}_s u = g \text{ in } \ \mathbb{R}^N \setminus \Omega. \tag{2.7}$$

What is a weak solution of (2.7)? When is (2.7) well-posed?

## Another fractional order Sobolev space

Let $g \in L^1(\mathbb{R}^N \setminus \Omega)$ be fixed and let

$$W_\Omega^{s,2} := \Big\{ u \in L^2(\Omega),\ |g|^{\frac{1}{2}} u \in L^2(\mathbb{R}^N \setminus \Omega), \int\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}\, dxdy < \infty \Big\}$$

be endowed with the norm

$$\|u\|^2_{W_\Omega^{s,2}} := \int_\Omega |u|^2\, dx + \int_{\mathbb{R}^N \setminus \Omega} |g||u|^2\, dx + \int\int_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}\, dxdy.$$

Then $W_\Omega^{s,2}$ is a Hilbert space (Ros-Oton and Valdinoci, JEMS 2017).

## Weak solutions of the Neumann problem

A $u \in W_{\Omega}^{s,2}$ is said to be a weak solution of (2.7) if for all $v \in W_{\Omega}^{s,2}$,

$$\int\int_{\mathbb{R}^{2N}\setminus(\mathbb{R}^N\setminus\Omega)^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy = \int_{\Omega} fv\,dx + \int_{\mathbb{R}^N\setminus\Omega} gv\,dx.$$

## Well-posedness of the Neumann problem

Let $f \in L^2(\Omega)$ and $g \in L^1(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N \setminus \Omega)$. Then the Neumann problem (2.7) has a weak solution if and only if

$$\int_{\Omega} f\,dx + \int_{\mathbb{R}^N\setminus\Omega} g\,dx = 0.$$

In that case, solutions are unique up to an additive constant.

## The Riemann Liouville fractional derivative

Let $\alpha \in (0,1)$ and define $g_\alpha(t) := \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases}$
It will be convenient to write $g_0 := \delta_0$, the Dirac measure concentrated at 0. Let $T > 0$ and $u \in C[0,T]$, with $g_{1-\alpha} * u \in W^{1,1}(0,T)$. The Riemann-Liouville fractional derivative of order $\alpha$ is defined by

$$D_t^\alpha u(t) := \frac{d}{dt}\Big(g_{1-\alpha} * u\Big)(t) = \frac{d}{dt}\int_0^t g_{1-\alpha}(t-\tau)\,u(\tau)\,d\tau.$$

## Properties of the Riemann Liouville fractional derivative

Let $0 < \alpha < 1$. Then the following assertions hold.

- $D_t^\alpha 1 = \frac{d}{dt}\left(g_{1-\alpha} * 1\right)(t) = \frac{d}{dt}\left(g_{2-\alpha}\right)(t) = g_{1-\alpha}(t) \neq 0.$
- $D_t^\alpha g_\alpha(t) = \frac{d}{dt}\left(g_{1-\alpha} * g_\alpha\right)(t) = \frac{d}{dt}\left(g_1\right)(t) = 0.$
- $D_t^1 u = \frac{d}{dt}\int_0^t g_0(t-\tau)u(\tau)\, d\tau = \frac{d}{dt}(u)(t) = u'(t).$

## The Caputo fractional derivative

The classical Caputo fractional derivative of order $0 < \alpha < 1$ is defined by

$$\mathbb{D}_t^{\alpha} u(t) = \Big(g_{1-\alpha} * u'\Big)(t) = \int_0^t g_{1-\alpha}(t-\tau) u'(\tau)\, d\tau. \qquad (2.8)$$

## Properties of the Caputo fractional derivative

- $\mathbb{D}_t^{\alpha} 1 = (g_{1-\alpha} * 0)(t) = 0.$
- $\mathbb{D}_t^{1} u(t) = \displaystyle\int_0^t g_0(t-\tau) u'(\tau)\, d\tau) = u'(t).$
- (Problem). One needs more regularity for $u$. More precisely, one also needs to know $u'(t)$ in order to calculate $\mathbb{D}_t^{\alpha} u(t)$ for $0 < \alpha < 1$.
- (2.8) is well-defined if and only if $u$ is absolutely continuous. (Gal and W., Springer Book series Mathematiques & Applications Vol 84, 2020).

## Modified Caputo derivative

We modify the fractional Caputo derivative as follows:

$$\partial_t^\alpha u(t) := D_t^\alpha\Big(u(t) - u(0)\Big) = \frac{d}{dt}\int_0^t g_{1-\alpha}(t-\tau)\,(u(\tau)-u(0))\,d\tau.$$

## Properties of the modified Caputo derivative

- $\partial_t^\alpha 1 = \dfrac{d}{dt}(g_{1-\alpha} * 0)(t) = 0.$
- $\partial_t^1 u(t) = \dfrac{d}{dt}\displaystyle\int_0^t g_0(t-\tau)(u(\tau)-u(0))\,d\tau) = \dfrac{d}{dt}(u(t)-u(0)) = u'(t).$
- (Advantage). One does not need more regularity for $u$, that is, one does not need to know $u'$ in order to calculate $\partial_t^\alpha u$ for $0 < \alpha < 1$.

## Some fractional in time ODEs

- The solution of $u'(t) = zu(t)$ ( $z \in \mathbb{C}$) is given by $u(t) = u(0)e^{tz}$.
- If $0 < \alpha \leq 1$, then the solution of $\mathbb{D}_t^\alpha u(t) = zu(t)$ is given by

$$u(t) = u(0)E_\alpha\left(zt^\beta\right).$$

where $E_\alpha$ is the Mittag-Leffler function defined for every $z \in \mathbb{C}$ by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}.$$

- It is clear that $E_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$.

# Outline

## Our control system

For $0 < \alpha \leq 1$, $0 < s \leq 1$ and $\omega \subset \Omega$ open, we consider the system

$$\begin{cases} \partial_t^\alpha u(t,x) + (-\Delta)^s u(t,x) = f|_\omega & \text{in } \ \Omega \times (0,T), \\ u = 0 \ \text{(EC)} & \text{in } \ (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(0,\cdot) = u_0 \ \text{(IC)} & \text{in } \ \Omega. \end{cases} \tag{3.1}$$

In (3.1), $f$ is the control function that is localized in a subset $\omega \subset \Omega$ and $u$ is the state to be controlled.

## Set of reachable states

The set of reachable states is given by

$$\mathcal{R}(u_0, T) = \left\{ u(\cdot, T) : \ f \in L^2(\omega \times (0,T)) \right\}.$$

## The three notions of controllability

1. We say that (3.1) is null controllable if there is a control function $f$ such that the solution $u$ satisfies $u(T,\cdot) = 0$ for some $T > 0$. This is equivalent to $0 \in \mathcal{R}(u_0, T)$.
2. The system (3.1) is said to be exactly controllable if for every $v \in L^2(\Omega)$ there is a control function $f$ such that the solution $u$ satisfies $u(T,\cdot) = v$ in $\Omega$. This is equivalent to $\mathcal{R}(u_0, T) = L^2(\Omega)$.
3. We say that (3.1) is approximately controllable if $\mathcal{R}(u_0, T)$ is dense in $L^2(\Omega)$.

## Equivalent of null and exact controllabilities

1. If the system is linear and reversible in time, then null and exact controllabilities are the same. This is the case for the heat equation.
2. For the wave equation, the two notions are different.

## Negative result for null controllability (Lü and Zuazua, Math. Control Signals Systems 2016)

- Let $0 < \alpha < 1$. Then the system (3.1) is never null controllable. That is, if $0 < \alpha < 1$, then there is no control function $f$ such that the solution $u$ can rest at some time $T > 0$.
- The same conclusion holds for any $\alpha \notin \mathbb{N}$.
- Solutions of (3.1) are represented in terms of Mittag-Leffler functions. The above negative result for the null controllability is essentially due to the behavior of the Mittag-Leffler functions with non-integer parameters $\alpha \notin \mathbb{N}$.

## Question

What happens if $\alpha \in \mathbb{N}$ and $0 < s < 1$? We will concentrate on the case $\alpha = 1$.

## Our control problem for $\alpha = 1$: The Schrödinger equation

Let $\Omega \subset \mathbb{R}^N$ be open, bounded and of class $C^{1,1}$. For $\omega \subset \Omega$ open, and $0 < s < 1$ we consider the following Schrödinger equation:

$$\begin{cases} i\partial_t u(t,x) + (-\Delta)^s u(t,x) = f\chi_\omega & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(0,\cdot) = u_0 & \text{in } \Omega. \end{cases} \tag{3.2}$$

- $f$ is the control function which is localized in $\omega \subset \Omega$.
- $u$ is the state to be controlled.

## Well-posedness

$\forall\, u_0 \in L^2(\Omega)$ and $f \in L^2((0,T)\times\Omega)$, the system (3.2) is well-posed as a Cauchy problem.

## The observability inequality (Biccari, 2016)

Let $\Gamma_0 := \{x \in \partial\Omega : \ (x \cdot \nu) > 0\}$ and $\omega := \mathcal{O} \cap \Omega$ where $\mathcal{O}$ is an open neighborhood of $\Gamma_0$ in $\mathbb{R}^N$. For $u_0 \in L^2(\Omega)$ and $f \in L^2((0,T) \times \Omega)$, let $u$ be the solution of (3.2). Then the following assertions hold.

- If $\frac{1}{2} < s < 1$, then for any $T > 0$ we have that

$$\|u_0\|^2_{L^2(\Omega)} \leq \int_0^T \int_\omega |u(t,x)|^2 \, dxdt. \tag{3.3}$$

- If $s = \frac{1}{2}$, then (3.3) holds for any $T > 2Pd(\Omega) =: T_0$, where $Pd(\Omega)$ is the Poincaré constant for the embedding $W_0^{s,2}(\overline{\Omega}) \hookrightarrow L^2(\Omega)$.
- If $0 < s < \frac{1}{2}$ such an inequality (3.3) does not hold.

### The null controllability for $\alpha = 1$ (Biccari, 2016)

Let $\Gamma_0 := \{x \in \partial\Omega : \ (x \cdot \nu) > 0\}$ and $\omega := \mathcal{O} \cap \Omega$ where $\mathcal{O}$ is an open neighborhood of $\Gamma_0$ in $\mathbb{R}^N$. For $u_0 \in L^2(\Omega)$ and $f \in L^2((0,T) \times \omega)$, let $u$ be the solution of (3.2). Then the following assertions hold.

- If $\frac{1}{2} < s < 1$, then there is a control function $f$ such that $u(T,\cdot) = 0$ in $\Omega$ for any $T > 0$.
- If $s = \frac{1}{2}$, then there is a control function $f$ such that $u(T,\cdot) = 0$ in $\Omega$ for any $T > T_0 := 2Pd(\Omega)$.
- If $0 < s < \frac{1}{2}$, then the system is not null controllable.

## Main ingredients used for the proof

The main tool needed to show the above obervability inequality and hence, the null controllability result is the following identity known as the fractional version of the Pohozaev identity. Let $\delta(x) := \mathrm{dist}(x, \partial\Omega)$, $u \in C^s(\mathbb{R}^N)$ and $u = 0$ in $\mathbb{R}^N \setminus \Omega$, be such that

- $u \in C^\beta(\Omega)$ for some $\beta \in [s, 1+2s]$.
- $\dfrac{u}{\delta^s} \in C^{0,\gamma}(\overline{\Omega})$ for some $0 < \gamma < 1$.
- $(-\Delta)^s u$ is pointwise bounded in $\Omega$.

Then the following identity holds.

$$\int_\Omega (-\Delta)^s u \,(x \cdot \nabla u)\; dx = \frac{2s-N}{2} \int_\Omega u(-\Delta)^s u \, dx - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu)\, d\sigma.$$

## The negative result for $0 < s < \frac{1}{2}$

The negative result is proved by direct computation with $\Omega = (-1, 1)$.

- In fact, one used the fact that the eigenvalues of $(-d_x^2)^s$ with zero Dirichlet exterior conditions are given by

$$\lambda_k = \left(\frac{k\pi}{2} - \frac{(2-2s)\pi}{8}\right)^{2s} + O\left(\frac{1}{k}\right) \quad \text{for} \quad k \geq 1. \tag{3.4}$$

- Using (3.4) one proves that

$$\lambda_{k+1} - \lambda_k \geq \gamma > 0 \iff s \geq \frac{1}{2}. \tag{3.5}$$

- Finally one uses (3.5) to show that the observability inequality does not if $0 < s < \frac{1}{2}$ and this implies that the system cannot be null controllable if $0 < s < \frac{1}{2}$.

# Outline

## The fractional heat equation (Interior control)

Let $0 < s < 1$ and consider the following system

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^s u(t,x) = f\chi_\omega & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(0,\cdot) = u_0, & \text{in } \Omega. \end{cases} \quad (4.1)$$

- If $N = 1$, then (4.1) is null controllable if and only if $\frac{1}{2} < s < 1$ (Biccari, W. and Zuazua, CPAA 2019).
- For every $N \geq 1$, (4.1) is approximately controllable (W., DCDS 2019).

## Boundary control problem for the classical heat equation

The classical boundary control problem for $\Delta$ is formulated as follows:

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = 0 & \text{in } \Omega \times (0,T), \\ Bu = f\chi_\omega & \text{on } \partial\Omega \times (0,T), \\ u(0,\cdot) = u_0, & \text{in } \Omega. \end{cases} \tag{4.2}$$

Here, $B$ is a boundary operator (Dirichlet, Neumann or Robin type), $u = u(t,x)$ is the state to be controlled, and $f = f(t,x)$ is the control function which is localized in a non-empty subset $\omega \subset \partial\Omega$.

## What about a boundary control with $(-\Delta)^s$?

Recall that we have mentioned above that the Dirichlet problem

$$(-\Delta)^s u = 0 \quad \text{in} \quad \Omega, \quad u = g \text{ on } \partial\Omega, \tag{4.3}$$

is not well-posed. Therefore we have the following situations.

- It follows from (4.3) that the system is not well-posed:

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^s u(t,x) = 0 & \text{in} \quad \Omega \times (0,T), \\ u = f\chi_\omega & \text{on} \quad \partial\Omega \times (0,T), \\ u(0,\cdot) = u_0 & \text{in} \quad \Omega. \end{cases}$$

- This shows that a boundary control does not make sense for the fractional Laplacian $(-\Delta)^s$ $(0 < s < 1)$. That is, the control function cannot be localized on a subset $\omega$ of $\partial\Omega$ (W., SICON 2019).

## What about boundary control with $(-\Delta)^s$? (W., SICON 2019)

The well-posed Dirichlet problem for $(-\Delta)^s$ is given by

$$(-\Delta)^s u = 0 \quad \text{in } \Omega, \quad u = g \text{ in } \mathbb{R}^N \setminus \Omega. \tag{4.4}$$

We have to the following situations.

- Since (4.4) is well posed, it follows that the system

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^s u(t,x) = 0 & \text{in } \Omega \times (0,T), \\ u = f\chi_\omega & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(0,\cdot) = u_0 & \text{in } \Omega, \end{cases} \tag{4.5}$$

  is well-posed as a Cauchy problem.

- Hence, the control MUST be localized in a subset $\omega \subset \mathbb{R}^N \setminus \Omega$.
- We shall call (4.5) an exterior control problem.

## What is so far known about the exterior control problem?

Given $u_0 \in L^2(\Omega)$, $0 < \alpha < 1$ and $\omega \subset \mathbb{R}^N \setminus \Omega$ an arbitrary non-empty open, we consider the system

$$\begin{cases} \partial_t^\alpha u(t,x) + (-\Delta)^s u(t,x) = 0 & \text{in } \ \Omega \times (0,T), \\ u = f\chi_\omega & \text{in } \ (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(0,\cdot) = u_0 & \text{in } \ \Omega. \end{cases} \tag{4.6}$$

Then for every $f \in L^2((0,T); W^{s,2}(\mathbb{R}^N \setminus \Omega))$, the system (4.6) is well-posed as a Cauchy problem.

## Explicit representation of solutions

Let $(-\Delta)^s_D$ be the realization in $L^2(\Omega)$ of $(-\Delta)^s$ with $u = 0$ in $\mathbb{R}^N \setminus \Omega$.

- $(-\Delta)^s_D$ has a compact resolvent.
- Let $(\varphi_n)_{n\in\mathbb{N}}$ be the normalized base of eigenfunctions of $(-\Delta)^s_D$.
- The unique solution $u$ of the system (4.6) is given by

$$u(t,x) = -\sum_{n=1}^{\infty}\left(\int_0^t \left(f(t-\tau,\cdot),\mathcal{N}_s\varphi_n\right)_{L^2(\mathbb{R}^N\setminus\Omega)}\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^{\alpha})\,d\tau\right)\varphi_n(x).$$

- Here, $E_{\alpha,\alpha}$ is the Mittag-Leffler given by

$$E_{\alpha,\alpha}(z) := \sum_{n=1}^{\infty}\frac{z^n}{\Gamma(\alpha n+\alpha)}, \quad z\in\mathbb{C}.$$

**An exterior controllability result (W., SICON 2019)**

Let $\omega \subset \mathbb{R}^N \setminus \Omega$ be an arbitrary non-empty open set. Then the system (4.6) is approximately controllable for any $T > 0$ and $f \in \mathcal{D}(\omega \times (0, T))$. That this,

$$\overline{\{u(\cdot, T) : \; f \in \mathcal{D}(\omega \times (0, T))\}}^{L^2(\Omega)} = L^2(\Omega).$$

## What is needed in the proof of the approximate controllability?

- We prove first the unique continuation property of the eigenvalues problem. That is, let $\lambda > 0$ and $\varphi \in W^{s,2}(\mathbb{R}^N)$ satisfy

$$(-\Delta)^s \varphi = \lambda \varphi \text{ in } \Omega \quad \text{and} \quad \varphi = 0 \text{ in } \mathbb{R}^N \setminus \Omega.$$

Let $\omega \subset \mathbb{R}^N \setminus \Omega$ be a non-empty open set. We have the following.

$$\text{If } \mathcal{N}_s \varphi = 0 \text{ in } \omega, \text{ then } \varphi = 0 \text{ in } \mathbb{R}^N. \tag{4.7}$$

- To prove (4.7) one uses the following.

$$\text{If } u = (-\Delta)^s u = 0 \text{ in } \omega, \text{ then } u = 0 \text{ in } \mathbb{R}^N. \tag{4.8}$$

- Notice that (4.8) does not make sense for a local operator like $\Delta$.

## What is needed in the proof of the approximate controllability?

- The dual system associated with the system (4.6) is given by

$$\begin{cases} D_{t,T}^{\alpha} v + (-\Delta)^s v = 0 & \text{in } (0,T)\times\Omega \\ v = 0 & \text{in } (0,T)\times(\mathbb{R}^N\setminus\Omega) \\ I_{t,T}^{1-\alpha} v(T,\cdot) = u_0 & \text{in } \Omega. \end{cases} \tag{4.9}$$

- Using some important tools of analytic functions we prove that the solution of (4.9) satisfies the unique continuation principle. That is, let $\omega \subset (\mathbb{R}^N \setminus \Omega)$ be an arbitrary non-empty open set.

$$\text{If } \mathcal{N}_s v = 0 \text{ in } (0,T)\times\mathcal{O}, \text{ then } v = 0 \text{ in } (0,T)\times\Omega. \tag{4.10}$$

- We obtain the approximate controllability as a direct consequence of the property (4.10).

**The fractional heat equation (exterior control) (W. and Zamorano, Control & Cybernetic 2020)**

Let $0 < s < 1$, $N = 1$ and consider the system

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^s u(t,x) = 0 & \text{in } \ \Omega \times (0,T), \\ u = f\chi_\omega & \text{in } \ (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(0,\cdot) = u_0, & \text{in } \ \Omega. \end{cases} \tag{4.11}$$

Then the system (4.11) is null controllable at any time $T > 0$ if and only if $\frac{1}{2} < s < 1$.

# Outline

## Open problem: The fractional heat equation (Interior control)

Let $0 < s < 1$ and consider the following system

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^s u(t,x) = f\chi_\omega & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(0,\cdot) = u_0, & \text{in } \Omega. \end{cases} \tag{5.1}$$

If $N \geq 2$, we still DO NOT know if (5.1) is null/exact controllable or not.

## Open problem: The fractional heat equation (exterior control)

Let $0 < s < 1$, $N \geq 1$ and consider the system

$$\begin{cases} \partial_t u(t,x) + (-\Delta)^s u(t,x) = 0 & \text{in } \Omega \times (0,T), \\ u = f\chi_\omega & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(0,\cdot) = u_0, & \text{in } \Omega. \end{cases} \tag{5.2}$$

- If $N \geq 2$, we still DO NOT know if (5.2) is null/exact controllable or not.
- If on replaces $u = f\chi_\omega$ with $\mathcal{N}_s u = f\chi_\omega$, even in dimension $N = 1$, we still DO NOT know if the system is null/exact controllable or not.

## Open problem: The fractional wave equation (Interior control)

Let $0 < s < 1$ and consider the following wave equation

$$\begin{cases} \partial_{tt} u(t,x) + (-\Delta)^s u(t,x) = f|_\omega & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(0,\cdot) = u_0, \quad u_t(0,\cdot) = u_1 & \text{in } \Omega. \end{cases}$$

- In any dimension $N \geq 1$, we dot not know if the system is null/exact controllable or not. We just know that it is approximately controllable (Louis-Rose and W., Applied Maths and Optimization 2020, to appear).
- If one replaces $u = 0$ by $\mathcal{N}_s u = 0$, then we still do not know much about the controllability properties of the system.

## STAY SAFE!

THANK YOU VERY MUCH AND STAY SAFE!