Fractional calculus: Anomalous diffusion. Generalized Pearson equations

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Abstract

In this essay, we study fractional calculus and its application to a real-world problem: anomalous diffusion. First, we present a review of the fundamentals of fractional calculus. Then, we analyze analytically and numerically the anomalous diffusion equation with drift, a partial differential equation that appears in some real-world applications related with anomalous super-dispersion. We show that the solution of the fractional diffusion equation with drift cannot be expressed in terms of elementary functions. Our numerical results show that the fractional term of the equation behaves as a mix of pure transport and pure diffusion, where the quantity of transport and diffusion respectively depends on the derivative order of the fractional term in a non-trivial way. Finally, we study the fractional Pearson equations, which constitute a new fractional analogue to the classical Pearson equations. In this way, we introduce fractional analogues of beta, gamma and normal distributions. Also, quasi-polynomials orthogonal with respect to these new distributions are presented and some conjectures about their zeros are stated.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Ariane Dores MANINTCHAP THUENTEU, 23 June 2015.

Contents

1. Introduction

Background

Fractional calculus is an extension of normal calculus where the order of derivatives is not an integer but a real number. Many known mathematicians have contributed to this theory over the years, among them Liouville, Riemann, Fourier, Leibniz and Grünwald.

Probably, the first application of fractional calculus was made by N. H. Abel in 1823 when he was studying the integral equation that arises in the formulation of the tautochrone problem. But we would also like to mention O. Heaviside, who in 1893 used fractional differential operators together with his famous operational calculus to study the *Age of the Earth* (see [24] or [26, Chapter 7]). For a long time, the theory of fractional calculus developed only as a pure theoretical field of mathematics. However, in the last decades, it was found that fractional derivatives and integrals provide a better tool to understand some physical phenomena, especially when processes with memory are considered [7]. Applications nowadays include modeling of viscoelastic and viscoplastic materials [41], chemical processes [40], and a wide range of engineering problems.

Aims

The main aims of this essay are to understand the principles of fractional calculus as well as to introduce new fractional analogues of some well-known results from classical analysis. Moreover, we consider some of its application on the real-world.

Overview of the thesis

To achieve these goals, we proceed as follows. In Chapter 2, basic definitions and notations are introduced. We pay special attention to those related with fractional calculus which will be used within the essay. In particular, we introduce Riemann-Liouville and Caputo fractional derivatives, which are considered in Chapter 3 and Chapter 4, respectively.

In Chapter 3, a real-world application of fractional calculus is analyzed in detail, namely super-dispersion that was observed in a ground water dispersion experiment [21]. We analyze the corresponding partial differential equation which models anomalous diffusion with drift. We study the equation analytically and numerically by means of the Fourier transform and a numerical scheme applied to a number of test cases, respectively.

In Chapter 4, we consider a new mathematical problem: to introduce fractional analogues of the well-known beta, gamma and normal distributions. In doing so, we consider fractional differential equations for which we provide analytical solutions. These new fractional analogues suggest us to introduce weighted orthogonal quasi-polynomials. Some conjectures about the zeros of these quasi-polynomials orthogonal with respect to the fractional analogues of the beta and gamma distribution are stated.

Finally, we conclude the work in Chapter 5 and give some recommendations and perspectives.

2. Fractional calculus

In this chapter, we present the notation and basic definitions that are used in the essay.

2.1 Basic ideas

It is possible to introduce fractional calculus from a standard result from classical differential and integral calculus [33, Theorem 6.18]

2.1.1 Theorem (Fundamental Theorem of Classical Calculus)**.** *Let $f : [a, b] \to \mathbf{R}$ be a continuous function, and let $F : [a, b] \to \mathbf{R}$ be defined by*

$$F(x) = \int_a^x f(t)dt.$$

Then, F is differentiable and

$$F' = f.$$

This theorem provides a close relation between differential operators and integral operators. One of the main objectives is to introduce new operators that retain this relation in a suitably generalized sense.

2.1.2 Definition.

- By D, we denote the operator that maps a differentiable function onto its derivative, i.e.

$$Df(x) := f'(x).$$

- By J_a, we denote the operator that maps a function, assumed to be (Riemann) integrable on the compact interval $[a, b]$, onto its primitive centered at a, i.e.

$$J_a f(x) := \int_a^x f(t)dt,$$

  for $a \leq x \leq b$.

- For $n \in \mathbf{N}$ we use the symbols D^n and J_a^n to denote the n-fold iterates of D and J_a, respectively, that is

$$D^1 := D, \quad J_a^1 := J_a, \quad D^n := DD^{n-1}, \quad J_a^n := J_a J_a^{n-1}, \quad n \geq 2.$$

From Theorem 2.1.1 it follows that

$$DJ_a f = f,$$

and therefore

$$D^n J_a^n f = f,$$

for $n \in \mathbf{N}$. Thus, D^n is the left inverse operator of the operator J_a^n in a suitable space of functions [6]. The generalization of this property to the case of not integer numbers is not straightforward.

Let us recall the following result [35, Eq. (2.16)] for the integral operator J_a^n.

2.1.3 Lemma. Let f be a Riemann integrable function on $[a, b]$. Then, for $a < x < b$ and $n \in \mathbf{N}$ the following relation holds

$$J_a^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt.$$

Moreover, from reference [6, p. 8], we have:

2.1.4 Lemma. Let $n, m \in \mathbf{N}$ such that $m > n$, and let f be a function having a continuous nth derivative on the interval $[a, b]$. Then,

$$D^n f = D^m J_a^{m-n} f.$$

2.2 Some special functions that appear in fractional calculus

In this section, a compilation of the special functions arising within the text is provided. For each function, formal definitions and main properties are summarized. Further details can be found in the basic references [1, 8, 9, 10, 29, 30, 35].

2.2.1 Pochhammer symbol. Let $z \in \mathbf{C}$ and $n \in \mathbf{N}$. The Pochhammer symbol $(z)_n$ with integer order n is defined by

$$(z)_0 := 1, \qquad (z)_n = z(z+1)\cdots(z+n-1), \quad n = 1, 2, \ldots.$$

The following relations hold

$$(1)_n = n!, \qquad (z)_n = (-1)^n (1-n-z)_n.$$

2.2.2 Binomial coefficients. Let $\alpha \in \mathbf{C}$ and $n \in \mathbf{N}$. The binomial coefficients are defined by

$$\binom{\alpha}{n} = \frac{(-1)^n(-\alpha)_n}{n!}.$$

In particular, if $\alpha = m$ and $m = 1, 2, \ldots$ is a positive integer, we have

$$\binom{m}{n} = \begin{cases} \dfrac{m!}{n!\,(m-n)!}, & m \geq n \geq 0, \quad n = 0, 1, \ldots, m \\ 0, & 0 \leq m < n. \end{cases}$$

The binomial coefficients can be also defined in the case of arbitrary complex β and $\alpha \neq -1, -2, \ldots$ as

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\,\Gamma(\alpha-\beta+1)}.$$

2.2.3 The gamma function $\Gamma(z)$. Let $z \in \mathbf{C}$. The Euler integral of the second kind

$$\Gamma(z) = \int_0^\infty x^{z-1} \exp(-x)dx, \quad \Re(z) > 0,$$

is referred to as the gamma function, where $\Re(z)$ denotes the real part of z. The following recurrence relation can be obtained from the definition of the gamma function by integration by parts

$$\Gamma(z+1) = z\,\Gamma(z), \qquad \Re(z) > 0.$$

Moreover,

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z)_n}$$

is a relation between the gamma function and the Pochhammer symbol.

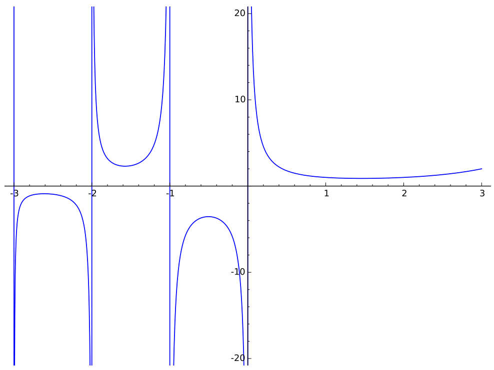

Figure 2.1: Graph of the gamma function $\Gamma(z)$ for $z \in [-3, 3]$. The gamma function has a pole at each negative integer.

2.2.4 The beta function $B(z, w)$. The Euler integral of the first kind

$$B(z,w) = \int_0^1 x^{z-1} (1-x)^{w-1} dx, \qquad \Re(z) > 0, \quad \Re(w) > 0,$$

is called the beta function. The following relation connects the beta function with the gamma function

$$B(z,w) = \frac{\Gamma(z)\,\Gamma(w)}{\Gamma(z+w)}.$$

2.2.5 The Gauss hypergeometric function. Let us define

$${}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| \, z \right) = \sum_{n=0}^{\infty} \frac{(a)_n \, (b)_n}{(c)_n} \frac{z^n}{n!}.$$

The parameters a, b, and c and the variable z may be complex and in order to avoid division by zero $c \neq 0, -1, -2, \ldots$. The series is convergent for $|z| < 1$ and for $|z| = 1$, $\Re(c - a - b) > 0$.

2.2.6 The Mittag-Leffler functions. Let us consider the following entire function (Mittag-Leffler function) defined by the series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \alpha > 0. \tag{2.2.1}$$

The more general series

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha > 0, \quad \beta > 0,$$

is also referred to as the Mittag-Leffler function. Obviously,

$$E_{\alpha,1}(z) = E_\alpha(z).$$

The Mittag-Leffler function is an important function in the field of fractional calculus. Like the exponential function result from the solution of integer order differential equations, the Mittag-Leffler function plays same role in the solution of non-integer order differential equations. For $\alpha = 1$, the Mittag-Leffler function is the exponential function.

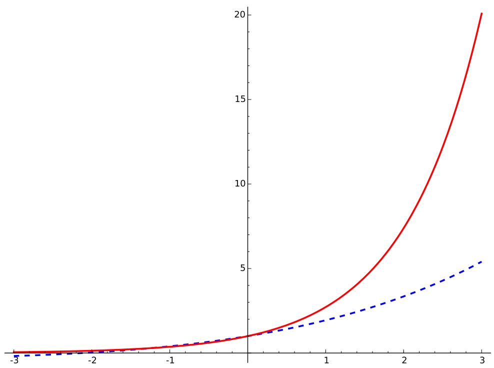

Figure 2.2: Graph of the Mittag-Leffler function $E_\alpha(z)$ with $z \in [-3,3]$ for $\alpha = 1$ (solid red) and $\alpha = 3/2$ (dashed blue).

2.3 Riemann-Liouville differential and integral operators

In this chapter, a first definition for fractional integral and differential operators J_a^α and D^α, $\alpha \notin \mathbf{N}$ is given.

2.3.1 Riemann-Liouville integrals.

2.3.2 Definition. Let $\alpha > 0$. The operator J_a^α defined on $\mathcal{L}_1[a,b]$ by

$$J_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt,$$

for $a \leq x \leq b$ is called the Riemann-Liouville fractional integral operator of order α.

It is obvious that for $\alpha = 0$, then J_a^0 is the identity operator I. Moreover, for $\alpha \in \mathbf{N}$ then the Riemann-Liouville fractional integral coincides with the classical definition of J_a^n in the case $n \in \mathbf{N}$ (notice that we have extended the domain from Riemann integrable functions to Lebesgue integrable functions).

Furthermore, if $\alpha > 1$, it is evident that the integral $J_a^\alpha f(x)$ exists for every $x \in [a,b]$: the integrand is the product of an integrable function $f(t)$ and the continuous function $g(t) = (x-t)^{\alpha-1}$. On the other hand, for $0 < \alpha < 1$ we need to recall the following result from reference [6, p. 13]

2.3.3 Theorem. *Let $f \in \mathcal{L}_1[a,b]$ and $\alpha > 0$. Then, the integral $J_a^\alpha f(x)$ exists for almost every $x \in [a,b]$. Moreover, the function $J_a^\alpha f$ itself is also an element of $\mathcal{L}_1[a,b]$.*

We also have:

2.3.4 Theorem. *Let $\alpha, \beta > 0$ and $f \in \mathcal{L}_1[a,b]$. Then,*

$$J_a^\alpha J_a^\beta f = J_a^{\alpha+\beta} f = J_a^\beta J_a^\alpha f$$

holds almost everywhere on $[a,b]$. If additionally $f \in \mathcal{C}[a,b]$ or $\alpha + \beta \geq 1$, then the identity holds everywhere on $[a,b]$.

Under certain assumptions, it is possible to interchange the limit operation and fractional integration [6, p. 21].

2.3.5 Theorem. *Let $\alpha > 0$. Assume that $\{f_k\}_{k=1}^{\infty}$ is a uniformly convergent sequence of continuous functions on $[a, b]$. Then we may interchange the fractional integral operator and the limit process,*

$$\left(J_a^{\alpha} \lim_{k\to\infty} f_k\right)(x) = \left(\lim_{k\to\infty} J_a^{\alpha} f_k\right)(x).$$

In particular, the sequence of functions $\{J_a^{\alpha} f_k\}_{k=1}^{\infty}$ is uniformly convergent.

2.4 Riemann-Liouville derivatives

In the classical situation, we have stated the following fundamental property in Lemma 2.1.4:

$$D^n f = D^m J_a^{m-n} f$$

where $m, n \in \mathbf{N}$ satisfy $m > n$.

Following reference [16, p. 70], we have:

2.4.1 Definition. Let $\alpha > 0$ and $m = \lceil \alpha \rceil$. The operator D_a^{α} defined by

$$D_a^{\alpha} f := D^m J_a^{m-\alpha} f$$

is called the Riemann-Liouville fractional differential operator of order α. Here, $\lceil \cdot \rceil$ denotes the ceiling function.

If $\alpha = 0$, we set $D_a^0 := I$ the identity operator. Similarly to the fractional integral operator, if $\alpha = n \in \mathbf{N}$ then D_a^n coincides with the classical differential operator. Moreover, from reference [16, Eq. (2.1.5)], we have:

$$(D_a^{\alpha} f)(x) = \frac{1}{\Gamma(m-\alpha)} D^m \int_a^x \frac{f(t)}{(x-t)^{\alpha-m+1}} dt.$$

In particular, if $0 < \alpha < 1$, then

$$(D_a^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\alpha}} dt.$$

Similarly to the fractional integral operator, the fractional differential operators form a semigroup.

2.4.2 Theorem. *Assume that $\alpha, \beta > 0$. Moreover, let $\phi \in \mathcal{L}_1[a, b]$ and $f = J_a^{\alpha+\beta}\phi$. Then,*

$$D_a^{\alpha} D_a^{\beta} f = D_a^{\alpha+\beta} f.$$

Moreover, from reference [6, p. 30], we have:

2.4.3 Theorem. *Let $\alpha \geq 0$. Then, for every $f \in \mathcal{L}_1[a, b]$*

$$D_a^{\alpha} J_a^{\alpha} f = f$$

almost everywhere.

For uniform convergent sequences the following result holds true.

2.4.4 Theorem. *Let $\alpha > 0$. Assume that $\{f_k\}_{k=1}^{\infty}$ is a uniformly convergent sequence of continuous functions on $[a, b]$, and that $D_a^{\alpha} f_k$ exists for every k. Moreover, assume that $\{D_a^{\alpha} f_k\}_{k=1}^{\infty}$ converges uniformly on $[a + \epsilon, b]$ for every $\epsilon > 0$. Then, for every $x \in (a, b]$ we have*

$$\left(\lim_{k\to\infty} D_a^{\alpha} f_k\right)(x) = \left(D_a^{\alpha} \lim_{k\to\infty} f_k\right)(x).$$

2.5 Relations between Riemann-Liouville integrals and derivatives

As we have defined, the fractional integral of order $\alpha > 0$ of a given function $f\colon \mathbf{R} \to \mathbf{R}$ is defined, for $t > 0$, as [16, 28]

$$I^{\alpha} f\,(t) = J_0^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds. \tag{2.5.1}$$

Note that for $\alpha < 1$ the integral may be singular, but it is well defined if, for example, $f \in \mathcal{L}^1_{\mathsf{loc}}(\mathbf{R})$.

Using (2.5.1), we have defined the fractional Riemann-Liouville derivative of order $\alpha \in (0,1)$, of f as [16, 28]

$${}^{\mathrm{RL}}D^{\alpha} f\,(t) = D_0^{\alpha} f(t) = D^1\left(I^{1-\alpha} f\right)\,(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds,$$

provided that the right hand side is defined for almost every $t \in \mathbf{R}^+$. This is well defined if, for example, f is absolutely continuous on every compact interval of $\mathbf{R}$.

Moreover, as in the integer case, we have

$${}^{\mathrm{RL}}D^{\alpha}(I^{\alpha} f)\,(t) = f(t);$$

but $I^{\alpha}({}^{\mathrm{RL}}D^{\alpha} f)$ is not, in general, equal to f. Indeed,

$$I^{\alpha}({}^{\mathrm{RL}}D^{\alpha} f)\,(t) = f(t) + c_1\, t^{\alpha-1},$$

where $c_1 \in \mathbf{R}$ (see reference [6, Theorem 2.23]).

2.6 Caputo derivatives

Now, we introduce the Caputo derivative, which is a variation of the Riemann-Liouville derivatives. It simplify computations in some situations, when we model real-world phenomena with fractional differential equations [6].

The Caputo fractional derivative [16, 28] can be defined in terms of the Riemann-Liouville fractional derivative as

$${}^{\mathrm{C}}D^{\alpha} f\,(t) = {}^{\mathrm{RL}}D^{\alpha} g\,(t),$$

with $g(t) = f(t) - f(0)$.

In addition, if f is an absolutely continuous function on every compact interval (of $\mathbf{R}$), we can write, for $\alpha \in (0,1)$,

$${}^{\mathrm{C}}D^{\alpha} f\,(t) = I^{1-\alpha} D^1 f\,(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\,ds. \tag{2.6.1}$$

Note also that if $\alpha \in (0,1)$ and f is a function for which the Caputo fractional derivative, ${}^{\mathrm{C}}D^{\alpha} f$, exists together with the Riemann-Liouville fractional derivative, ${}^{\mathrm{RL}}D^{\alpha} f$, both of them of order α, then we have the following relation [16, (2.4.8), p. 91]

$${}^{\mathrm{C}}D^{\alpha} f\,(t) = {}^{\mathrm{RL}}D^{\alpha} f\,(t) - \frac{f(0)}{\Gamma(1-\alpha)}\, t^{\alpha}.$$

2.7 Relation between fractional integrals and Caputo derivatives

As in the integer case, we have

$$^{\mathrm{C}}D^{\alpha}(I^{\alpha}f)\,(t) = f(t);$$

but $I^{\alpha}(^{c}D^{\alpha}f)$ is not, in general, equal to f. Indeed,

$$I^{\alpha}(^{c}D^{\alpha}f)\,(t) = f(t) + c_2,$$

where $c_2 \in \mathbf{R}$.

2.8 Relation between the Caputo fractional derivative and the Mittag-Leffler function

Let us recall the Mittag-Leffler function $E_{\alpha}(z)$ defined in (2.2.1), and let us consider the following fractional differential equation

$$^{c}D^{\alpha}y(x;\alpha) = y(x;\alpha), \tag{2.8.1}$$

which is a fractional analogue of the differential equation for the exponential function. Let us assume that there exists a solution of the above fractional differential equation with a formal power expansion as

$$y(x;\alpha) = \sum_{n=0}^{\infty} a_n(\alpha)x^{n\alpha}.$$

Using (2.6.1), we have

$$^{c}D^{\alpha}\,x^{n\alpha} = \frac{\Gamma(1+\alpha n)}{\Gamma(1+\alpha(n-1))}x^{\alpha(n-1)}.$$

Substitution into the fractional differential equation (2.8.1) yields

$$\sum_{n=0}^{\infty} a_{n+1}(\alpha)x^{n\alpha}\frac{\Gamma(1+\alpha(n+1))}{\Gamma(1+\alpha n)} = \sum_{n=0}^{\infty} a_n(\alpha)x^{n\alpha},$$

which implies

$$a_{n+1}(\alpha) = a_n(\alpha)\frac{\Gamma(1+\alpha n)}{\Gamma(1+\alpha(n+1))}.$$

The solution of the above recurrence relation is

$$a_n(\alpha) = \frac{a_0(\alpha)}{\Gamma(1+n\alpha)},$$

that is, the solution of the fractional differential equation (2.8.1) is up to a multiplicative constant the Mittag-Leffler function (2.2.1). Notice that in the limit as $\alpha \to 1^{-}$, we obtain

$$a_n = \frac{a_0}{n!}$$

which gives as result the exponential function $\exp(x)$ up to a multiplicative constant.

2.9 Fractional analogue of the logarithmic function

Next, we introduce a fractional elementary function which seems to be new in the literature. As it is well-known, the logarithmic function $y(x) = \log(1+x)$ is a solution of the following first order differential equation

$$y'(x) = \frac{1}{1+x}.$$

Let us now consider the following fractional differential equation

$$^{c}D^{\alpha} y(x;\alpha) = \frac{1}{1+x^{\alpha}}.$$

If we consider a formal power expansion of $y(x;\alpha)$ as

$$y(x;\alpha) = \sum_{n=0}^{\infty} a_n(\alpha)\, x^{n\alpha},$$

from the fractional differential equation we obtain

$$y(x;\alpha) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{\Gamma(\alpha(n-1)+1)}{\Gamma(n\alpha+1)} x^{n\alpha}.$$

Notice that the fractional derivative of $y(x;\alpha)$ is

$$\sum_{n=1}^{\infty} (-1)^{n-1} x^{\alpha(n-1)} = \frac{1}{1+x^{\alpha}},$$

in accordance with the fractional differential equation. Moreover, the limit as $\alpha \to 1^{-}$ of $y(x;\alpha)$ and of its fractional derivative is given by

$$\lim_{\alpha \to 1^{-}} y(x;\alpha) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \log(1+x),$$

and

$$\lim_{\alpha \to 1^{-}} {}^{c}D^{\alpha} y(x;\alpha) = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1} = \frac{1}{1+x}.$$

After this short introduction to fractional calculus, we shall show a real-world application in the next chapter.

3. Real-world application of fractional calculus

This chapter gives an example of a real-world application of fractional calculus. We first introduce the classical diffusion equation, and then the anomalous diffusion equation. We solve the latter analytically and numerically. Finally, we give some examples.

3.1 The diffusion equation

Consider a glass of water. The water consists of water molecules that move randomly and interact with each other. If, for example a drop of water-based ink is introduced in the water in a gentle way, the ink molecules will move similarly to the water and as a consequence the drop of ink will dissolve until the entire glass of water is equally colored. This process is referred to as diffusion: "the process by which matter is transported from one part of a system to another as a result of random molecular motions" [5, p. 1]. Thus, diffusion is a macroscopic interpretation of a microscopic phenomenon.

3.1.1 Definition (Diffusion equation)**.** Let $D > 0$. The diffusion equation is defined by

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} \tag{3.1.1}$$

where C is a function of x and t, $x \in \mathbf{R}$ and $t > 0$.

One approach to solve the diffusion equation, is to apply the Fourier transform in x to (3.1.1). We demonstrate this below by computing the Green's function of the equation. That is, we consider the initial condition $\widehat{C}(k,0) \equiv 1$. We have

$$\frac{d\widehat{C}(k,t)}{dt} = D(ik)^2\widehat{C}(k,t)$$

where $\widehat{C}$ is the Fourier transform of C and k is the frequency variable. Thus,

$$\begin{aligned}\frac{d\widehat{C}(k,t)}{dt} = D(ik)^2\widehat{C}(k,t) \quad &\Rightarrow \quad \frac{d\widehat{C}(k,t)}{\widehat{C}(k,t)} = D(ik)^2 dt \\ &\Rightarrow \quad \widehat{C}(k,t) = e^{D(ik)^2 t} \text{ by using the initial condition } \widehat{C}(k,0) \equiv 1 \\ &\Rightarrow \quad \widehat{C}(k,t) = e^{-Dtk^2}.\end{aligned}$$

By inverting the Fourier transform, we obtain the Green's function

$$C(x,t) = \frac{1}{2\sqrt{\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right).$$

In case of a boundary value problem the final expression for $C(x,t)$ may differ since initial conditions must be considered.

3.2 The anomalous diffusion equation

In certain cases, the normal diffusion equation is not a satisfactory model for the observed data. For example, in a study on ground water dispersion, higher diffusion speeds than normal were observed [21].

Normal diffusion models were not able to model this phenomenon. In contrast, a fractional diffusion model based on the fractional diffusion equation with drift, was able to model the data and explain this observation of so-called super-diffusion. In this section, we study the fractional diffusion equation with drift. We start our study on the microscopic level.

Let $X, X_1, \ldots, X_n,\ n \in \mathbf{N}^*$ be independent and identically distributed random variables. (These variables could for example model the movement of the water molecules in the example above.) Suppose that the probability $P_r\{X > x\} = Ax^{-\alpha}$, where $A > 0$ and $1 < \alpha < 2$. We have $P_r\{X \leq x\} = 1 - Ax^{-\alpha}$ and $P_r\{X \leq x\} = 0 \Rightarrow x = A^{1/\alpha}$. So the cumulative distribution function of X is

$$F_X(x) = \begin{cases} 0, & x < A^{1/\alpha}, \\ 1 - Ax^{-\alpha}, & x \geq A^{1/\alpha}. \end{cases}$$

Its probability density function is

$$f_X(x) = \begin{cases} 0, & x < A^{1/\alpha}, \\ A\alpha x^{-\alpha-1}, & x \geq A^{1/\alpha}. \end{cases}$$

Its mean is

$$\mu = E[X] = \int_{A^{1/\alpha}}^{\infty} x f_X(x)\ dx = \int_{A^{1/\alpha}}^{\infty} x A\alpha x^{-\alpha-1}\ dx = A\alpha \left[\frac{1}{-\alpha+1} x^{-\alpha+1}\right]_{A^{1/\alpha}}^{\infty} = \frac{\alpha}{\alpha-1} A^{1/\alpha}.$$

Its second order moment is

$$E[X^2] = \int_{A^{1/\alpha}}^{\infty} x^2 f_X(x)\ dx = \int_{A^{1/\alpha}}^{\infty} x^2 A\alpha x^{-\alpha-1}\ dx = A\alpha \left[\frac{1}{-\alpha+2} x^{-\alpha+2}\right]_{A^{1/\alpha}}^{\infty} = \infty,$$

since $0 < -\alpha + 2 < 1$. We have $\mu < \infty$, but $E[X^2] = \infty$. So, the extended central limit theorem in the reference [23] implies that

$$\lim_{n \to +\infty} P_r \left\{ \frac{X_1 + \cdots + X_n - n\mu}{n^{1/\alpha}} \leq x \right\} \to P_r\{Z_\alpha \leq x\}, \tag{3.2.1}$$

where Z_α is a stable random variable with index α, whose density $g_\alpha(x)$ has Fourier transform

$$\widehat{g}_\alpha(k) = \int_{-\infty}^{\infty} e^{-ikx} g_\alpha(x)\ dx = \exp\left(D(ik)^\alpha\right), \tag{3.2.2}$$

for some $D > 0$ depending on A and α.

To simulate X, we use the inverse cumulative distribution method which says that if X is a random variable with a cumulative distribution function F_X, then $X = F_X^{-1}(U)$, where U is uniform on $(0, 1)$. Let us set $y = F_X(x) = 1 - Ax^{-\alpha}$, then $x = F_X^{-1}(y) = \left(\frac{A}{1-y}\right)^{1/\alpha}$. So $X = \left(\frac{A}{1-U}\right)^{1/\alpha}$, where U is uniform on $(0, 1)$.

Now, let us define $Y_i = v\Delta t + (\Delta t)^{1/\alpha} X_i$ where $\Delta t = t/n$, $t > 0$,

$$X_i = \left(\frac{A}{1-U_i}\right)^{1/\alpha} - A^{1/\alpha}\frac{\alpha}{\alpha-1},$$

and $U_1, \ldots, U_n$ are independent uniform $(0,1)$ random variables. Since $E[X] = \mu = A^{1/\alpha}\frac{\alpha}{\alpha-1}$, $E[X^2] = \infty$ and $U_1, \ldots, U_n$ are independent uniform $(0,1)$ random variables, then $X_1, \ldots, X_n$ are independent random variables with mean 0 and infinite variance. Then, the extended central limit theorem (3.2.1) implies that $X_1 + \cdots + X_n \approx n^{1/\alpha} Z_\alpha$ for n large. So we have

$$\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n}\left(v\Delta t + (\Delta t)^{1/\alpha} X_i\right) = vn\Delta t + (\Delta t)^{1/\alpha}\sum_{i=1}^{n} X_i \approx vn\Delta t + (\Delta t)^{1/\alpha} n^{1/\alpha} Z_\alpha$$
$$= vn\Delta t + (n\Delta t)^{1/\alpha} Z_\alpha = vt + t^{1/\alpha} Z_\alpha,$$

by using $t = n\Delta t$. So $Y_1 + \cdots + Y_n \approx vt + t^{1/\alpha} Z_\alpha$, where Z_α is a stable random variable with index α, whose density $g_\alpha(x)$ has Fourier transform (3.2.2), and distribution function

$$G_\alpha(x) = \int_{-\infty}^{x} g_\alpha(u)\, du.$$

We have

$$P_r\{vt + t^{1/\alpha} Z_\alpha \le x\} = P_r\left\{Z_\alpha \le \frac{x - vt}{t^{1/\alpha}}\right\} = \int_{-\infty}^{t^{-1/\alpha}(x-vt)} g_\alpha(u)\, du$$
$$= \int_{-\infty}^{x} g_\alpha(t^{-1/\alpha}(y - vt)) t^{-1/\alpha} dy,$$

using the subtitution $u = t^{-1/\alpha}(y - vt)$. Hence the distribution function of the limit $Y = vt + t^{1/\alpha} Z_\alpha$ is

$$F_Y(x) = \int_{-\infty}^{x} g_\alpha(t^{-1/\alpha}(y - vt)) t^{-1/\alpha} dy.$$

Let us find the Fourier transform of its density

$$h_\alpha(x) = g_\alpha(t^{-1/\alpha}(x - vt)) t^{-1/\alpha}, \tag{3.2.3}$$

by using direct computations

$$\begin{aligned}
\widehat{h}_\alpha(k) &= \int_{-\infty}^{\infty} e^{-ikx} g_\alpha(t^{-1/\alpha}(x - vt)) t^{-1/\alpha}\, dx \\
&= \int_{-\infty}^{\infty} e^{-ik(vt + t^{1/\alpha}u)} g_\alpha(u) t^{-1/\alpha} t^{1/\alpha}\, du \quad \text{using the subtitution } u = t^{-1/\alpha}(x - vt) \\
&= e^{-ikvt}\int_{-\infty}^{\infty} e^{-ikt^{1/\alpha}} g_\alpha(u)\, du \\
&= e^{-ikvt}\, e^{D(ikt^{1/\alpha})^\alpha} \quad \text{by using (3.2.2)} \\
&= e^{-ikvt + Dt(ik)^\alpha}.
\end{aligned}$$

Let us define the fractional diffusion equation with drift:

3.2.1 Definition (Fractional diffusion equation with drift)**.** Let $v > 0$, $D > 0$ and $1 < \alpha < 2$. The fractional diffusion equation with drift is

$$\frac{\partial C}{\partial t} = -v\frac{\partial C}{\partial x} + D\frac{\partial^\alpha C}{\partial x^\alpha}, \tag{3.2.4}$$

where C is a function of x and t, $x \in \mathbf{R}$ and $t > 0$, v is given in meters per second and the fractional diffusion/dispersion coefficient D is given in meters$^\alpha$ per second. Notice that in the case $\alpha = 2$ then D is given in meters^2 per second.

3.2.2 Remark. The function h_α defined in (3.2.3) solves the fractional diffusion equation with drift. Since $x \in \mathbf{R}$, then, by taking the Fourier transform in x of (3.2.4), we have:

$$\frac{d\widehat{C}}{dt} = -v(ik)\widehat{C} + D(ik)^\alpha \widehat{C},$$

and

$$\begin{aligned}
\frac{d\widehat{C}}{dt} = -v(ik)\widehat{C} + D(ik)^\alpha \widehat{C} \quad &\Rightarrow \quad \frac{d\widehat{C}}{\widehat{C}} = (-vik + D(ik)^\alpha)dt \\
&\Rightarrow \quad \ln \widehat{C} = (-vik + D(ik)^\alpha)t + \text{constant} \\
&\Rightarrow \quad \widehat{C}(k,t) = e^{-vikt + Dt(ik)^\alpha}, \text{ by using the initial condition } \widehat{C}(k,0) \equiv 1 \\
&\Rightarrow \quad \widehat{C}(k,t) = \widehat{h}_\alpha(k).
\end{aligned}$$

The main difficulty of this approach is to find the inverse Fourier transform of $\widehat{h}_\alpha(k)$, which might involve cumbersome computations. In the case $\alpha = 2$, we recover the classical situation and it can be obtained by integration [23, p. 5] as

$$C(x,t) = \frac{\exp\left(-\frac{(x-tv)^2}{4D\,t}\right)}{2\sqrt{\pi}\sqrt{D\,t}},$$

but for $1 < \alpha < 2$, the result cannot be expressed in terms of elementary functions. In case of a boundary value problem, the final expression for $C(x,t)$ may differ.

Another way to solve the fractional diffusion equation with drift is to proceed numerically as described below.

3.2.3 Numerical solution of the anomalous diffusion equation. After solving (3.2.4) analytically, we solve it numerically in this section by considering [22, theorem 2.7] as follows:

3.2.4 Theorem. *The implicit Euler method solution to* (3.2.4) *with* $1 < \alpha \leq 2$ *on the finite domain* $L \leq x \leq R$, *with boundary conditions* $C(x = L, t) = 0$ *and* $C(x = R, t) = b_R(t)$ *for all* $t \geq 0$, *based on the shifted Grünwald approximation*

$$\frac{\partial^\alpha C(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(-\alpha)} \lim_{M\to\infty} \frac{1}{h^\alpha} \sum_{k=0}^{M} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} C(x-(k-1)h, t), \tag{3.2.5}$$

where $h = \frac{(x-L)}{M}$, *is consistent and unconditionally stable.*

The proof can be found in reference [22].

Let us consider the following fractional advection-dispersion flow equation

$$\frac{\partial C}{\partial t} = -v\frac{\partial C}{\partial x} + D\frac{\partial^\alpha C}{\partial x^\alpha}, \qquad (x,t) \in [L,R] \times [0,T] \tag{3.2.6}$$

with the initial and boundary conditions:

$$\begin{cases} C(x,0) = F(x), & L < x < R, \\ C(L,t) = 0, \quad \dfrac{\partial C(R,t)}{\partial t} = 0, & 0 \leq t \leq T, \end{cases} \tag{3.2.7}$$

where $F(L) = 0$ and $F'(R) = 0$. Theorem 3.2.4 is also true on these boundary conditions. We consider an equally spaced mesh of $M+1$ points for the spatial domain $L < x < R$, N constant time steps for the temporal domain $0 \leq t \leq T$. We denote the spatial grid points by $x_i = L + ih,\ i = 0, 1, \ldots, M$ and the temporal grid points by $t_n = n\tau,\ i = 0, 1, \ldots, N$. Here, the grid spacing is simply $h = \frac{R-L}{M}$ in the spatial domain and $\tau = \frac{T}{N}$ in the temporal domain.

At the grid point (x_i, t_n), (3.2.6) becomes

$$\left.\frac{\partial C}{\partial t}\right|_{(x_i,t_n)} = -v \left.\frac{\partial C}{\partial x}\right|_{(x_i,t_n)} + D \left.\frac{\partial^\alpha C}{\partial x^\alpha}\right|_{(x_i,t_n)} \tag{3.2.8}$$

The time derivative on the left hand side of (3.2.8) can be approximated by

$$\left.\frac{\partial C}{\partial t}\right|_{(x_i,t_n)} \approx \frac{C_i^{n+1} - C_i^n}{\tau},$$

and the spatial derivative on the right hand side of (3.2.8) can be approximated by

$$\left.\frac{\partial C}{\partial x}\right|_{(x_i,t_n)} \approx \frac{C_i^{n+1} - C_{i-1}^{n+1}}{h},$$

that is, we exploit an up-wind scheme (since $v > 0$). By using (3.2.5), the spatial fractional derivative on the right hand side of (3.2.8) can be approximated by

$$\left.\frac{\partial^\alpha C}{\partial x^\alpha}\right|_{(x_i,t_n)} \approx \frac{1}{h^\alpha \Gamma(-\alpha)} \sum_{k=0}^{i+1} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} C_{i-k+1}^{n+1}.$$

As a consequence, (3.2.8) becomes

$$\frac{C_i^{n+1} - C_i^n}{\tau} = -v\frac{C_i^{n+1} - C_{i-1}^{n+1}}{h} + \frac{D}{h^\alpha}\sum_{k=0}^{i+1} g_k C_{i-k+1}^{n+1}, \quad \text{for} \quad i = 1, 2, \ldots, M-1, \tag{3.2.9}$$

where g_k is the normalized Grünwald weight defined in [22] as

$$g_k = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)},$$

with $g_0 = 1$ and $g_1 = -\alpha$. Let us define $E = v\frac{\tau}{h}$ and $B = D\frac{\tau}{h^\alpha}$. Then, (3.2.9) can be written as

$$C_i^{n+1} - C_i^n = -E(C_i^{n+1} - C_{i-1}^{n+1}) + B\sum_{k=0}^{i+1} g_k C_{i-k+1}^{n+1},$$

which can be arranged as

$$-g_0 B C_{i+1}^{n+1} + (1 + E - g_1 B)C_i^{n+1} - (E + g_2 B)C_{i-1}^{n+1} - B\sum_{k=0}^{i+1} g_k C_{i-k+1}^{n+1} = C_i^n. \tag{3.2.10}$$

So, (3.2.10) can be written in matrix form as $A\mathbf{C}^{n+1} = \mathbf{C}^n$, where

$$\mathbf{C}^{n+1} = (C_0^{n+1}, C_1^{n+1}, \ldots, C_M^{n+1})^T, \quad \mathbf{C}^n = (C_0^n, C_1^n, \ldots, C_M^n)^T$$

and A is a $(M+1)\times(M+1)$ matrix defined as follows: for $i=1,\ldots,M-1$ and $j=1,\ldots,M-1$,

$$A_{i,j}=\begin{cases}0, & \text{if} \quad j\geq i+2,\\ -g_0B, & \text{if} \quad j=i+1,\\ 1+E-g_1B, & \text{if} \quad j=i,\\ -E-g_2B, & \text{if} \quad j=i-1,\\ -g_{i-j+1}B, & \text{if} \quad j<i-1.\end{cases} \tag{3.2.11}$$

For $\alpha=2$, we approximate $\frac{\partial^2 C}{\partial x^2}\Big|_{(x_i,t_n)}$ by

$$\frac{\partial^2 C}{\partial x^2}\Big|_{(x_i,t_n)}\approx\frac{C_{i+1}^{n+1}-2C_i^{n+1}+C_{i-1}^{n+1}}{h^2}.$$

So (3.2.8) becomes

$$-BC_{i+1}^{n+1}+(1+E+2B)C_i^{n+1}-(E+B)C_{i-1}^{n+1}=C_i^n, \quad \text{for} \quad i=1,2,\ldots,M-1,$$

and the matrix A is written as

$$A_{i,j}=\begin{cases}0, & \text{if} \quad j\geq i+2,\\ -B, & \text{if} \quad j=i+1,\\ 1+E+2B, & \text{if} \quad j=i,\\ -E-B, & \text{if} \quad j=i-1,\\ 0, & \text{if} \quad j<i-1.\end{cases} \tag{3.2.12}$$

Similarly, for $\alpha=1$, the matrix A is written as

$$A_{i,j}=\begin{cases}0, & \text{if} \quad j\geq i+1,\\ 1+E-B, & \text{if} \quad j=i,\\ -E+B, & \text{if} \quad j=i-1,\\ 0, & \text{if} \quad j<i-1.\end{cases} \tag{3.2.13}$$

Boundary conditions:

- Since $C(x,0)=F(x)$ for all $L\leq x\leq R$, then $C_0^0=F(x_0)$, $C_1^0=F(x_1)$, $\ldots$, $C_M^0=F(x_M)$. So we shall initialize our matrix $\mathbf{C}$ as $\mathbf{C}^0=(F(x_0),F(x_1),\ldots,F(x_M))$.
- Since $C(L,t)=0$ for all $0\leq t\leq T$, Then $C_0^0=C_0^1=\cdots=C_0^N=0$. Hence $A_{0,0}=1$ and $A_{0,j}=0$, for $j=1,\ldots,M$.
- We have

$$\frac{\partial C}{\partial x}\Big|_{(x_i,t_n)}\approx\frac{C_i^{n+1}-C_{i-1}^{n+1}}{h} \quad \text{so} \quad \frac{\partial C}{\partial x}\Big|_{(x_M,t_n)}\approx\frac{C_M^{n+1}-C_{M-1}^{n+1}}{h}.$$

Since $\frac{\partial C(R,t)}{\partial x}=0$, then $C_M^{n+1}-C_{M-1}^{n+1}=0$ which implies $C_M^{n+1}=C_{M-1}^{n+1}$. Hence $A_{M,M}=1$, $A_{M,M-1}=-1$ and $A_{M,j}=0$ for $j=0,\ldots,M-2$.

A Python code for solving the anomalous diffusion equation with drift (3.2.6) with boundary conditions (3.2.7) has been included in Appendix A and used in our numerical experiments.

3.2.5 Explicit examples. We solve the anomalous diffusion equation with drift (3.2.6) with boundary conditions (3.2.7) by considering the numerical scheme described above in the domain $[0,1]\times[0,1]$ for some specific values of the parameters, namely: $v = 0.5$, $D = 0.1$, and two choices of $C(x,0) = F(x)$ in (3.2.7). In the first case we have assumed that the initial concentration is given by a normal distribution $F_1(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\left(\frac{x-\mu}{\sigma}\right)^2\right)$ (see Figure 3.1 below), with mean $\mu = 0.3$, standard deviation $\sigma = 0.05$, $f(0) = 0$ and $f'(1) = 0$. Moreover, we have also considered as initial concentration $F_2(x) = 1_{[0.05,1]}(x)$ (see Figure 3.2 below). In our numerical experiments we consider $M = 1000$ divisions in space, $N = 2.5M$ divisions in time and $C^0 = (f(x_0), f(x_1), \ldots, f(x_M))^T$. Also, the value of the fractional parameter α varies as $\alpha = 1,\ 1.25,\ 1.5,\ 1.75$, and 2. For the first case $\alpha = 1$, the matrix A is given by (3.2.13). For the second, third, and fourth cases the matrix A is given by (3.2.11). Finally for the last case $\alpha = 2$ –which corresponds to the classical non-fractional situation– the matrix A is given by (3.2.12).

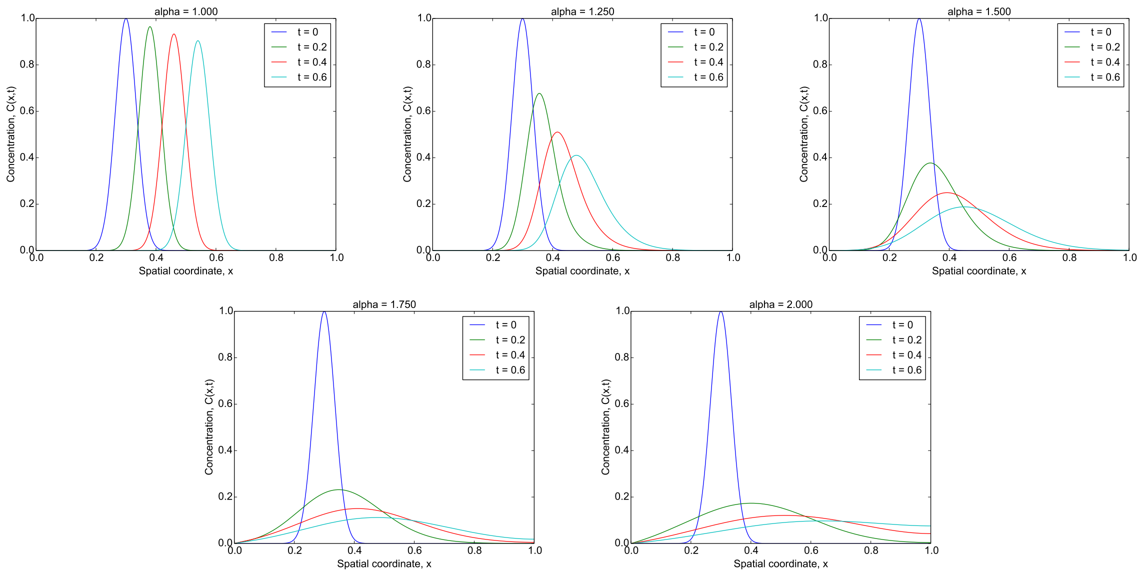

Figure 3.1: The concentration $C(x,t)$ as a function of x, solution of the anomalous diffusion equation with drift (3.2.6) with boundary conditions (3.2.7) in the particular case $C(x,0) = F(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\left(\frac{x-\mu}{\sigma}\right)^2\right)$ with $\mu = 0.3$ and $\sigma = 0.05$. The graphs show the solution $C(x,t)$ for $t = 0,\ 0.2,\ 0.4$ and 0.6 for several values of the fractional parameter α. From left to right and top to bottom, α varies from 1 to 2 as $\alpha = 1$, $\alpha = 1.25$, $\alpha = 1.5$, $\alpha = 1.75$, and $\alpha = 2$.

Figure 3.1 shows the result for the gaussian initial condition. For $\alpha = 1$, we have a pure transport equation. Thus the solution is a pure translation of the initial profile at the speed v. As shown in the graphs, the peak of concentration moves with the constant speed v as expected. However, we also see that the shape of the gaussian spreads out with time, which was not expected. This is due to diffusion introduced by the numerical scheme [3]. For $\alpha = 1.25, 1.5, 1.75$ and 2, the peak of concentration moves with time, but not exactly with the speed v. We can also see that the diffusion grows with α. So we can hypothesize that the fractional term behaves as a mix of pure transport and pure diffusion, where the quantity of transport and diffusion respectively depends on α in a non-trivial way.

Next, in Figure 3.2 shows the results for the step function initial condition. For $\alpha = 1$, we see that there is transport with the speed v as in the previous case. The smoothening of the concentration profile with time indicates that there is numerical dissipation. Moreover, no oscillations are visible. Thus, there is no numerical dispersion as expected for the up-wind scheme [3].

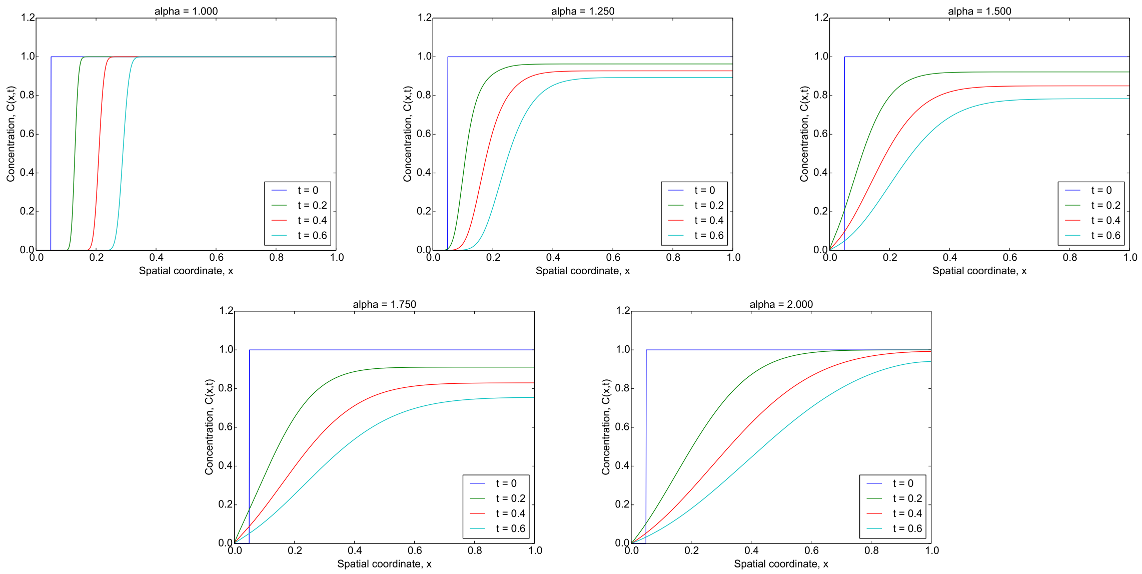

Figure 3.2: The concentration $C(x,t)$ as a function of x, solution of the anomalous diffusion equation with drift (3.2.6) with boundary conditions (3.2.7) in the particular case $C(x,0) = F(x) = 1_{[0.05,1]}(x)$. The graphs show the solution $C(x,t)$ for $t = 0$, 0.2, 0.4 and 0.6 for several values of the fractional parameter α. From left to right and top to bottom, α varies from 1 to 2 as $\alpha = 1$, $\alpha = 1.25$, $\alpha = 1.5$, $\alpha = 1.75$, and $\alpha = 2$.

In this chapter, we have analyzed and solved the fractional diffusion equation with drift which appears in anomalous diffusion. Another class of fractional differential equations (fractional Pearson equation) is presented in the next chapter. We exploit it to create fractional analogues of the gamma, beta and normal distributions which might be used in many areas like statistics, special functions, etc.

4. Generalized Pearson equations

4.1 Introduction

We define a Pearson density like in [15, p. 16] by a valid solution to the first order differential equation

$$\frac{1}{\varrho(x)}\varrho'(x) = -\frac{a_0 + x}{c_0 + c_1 x + c_2 x^2}. \tag{4.1.1}$$

Since the shape of the curve of solutions of (4.1.1) varies considerably, with the parameters a_0, c_0, c_1, and c_2, Pearson classified into a number of types (see the original [27] or a more recent presentation [15] for more explanations). Among these solutions, some are widely used in statistics: the normal distribution

$$\varrho_N(x) = \frac{\exp(-x^2)}{\sqrt{\pi}}, \qquad x \in \mathbf{R}, \tag{4.1.2}$$

the gamma distribution

$$\varrho_G(x;a) = \frac{x^a \exp(-x)}{\Gamma(a+1)}, \qquad a > -1, \quad x > 0, \tag{4.1.3}$$

and the beta distribution

$$\varrho_B(x;a,b) = \frac{x^a\,(1-x)^b\,\Gamma(a+b+2)}{\Gamma(a+1)\,\Gamma(b+1)}, \qquad a,b > -1, \quad x \in (-1,1). \tag{4.1.4}$$

They are weight functions on I [34], where I is the corresponding support for each of them. Let us write the Pearson equation for the weight function of orthogonality $\varrho(x)$

$$\frac{d}{dx}(\sigma(x)\varrho(x)) = \tau(x)\varrho(x), \tag{4.1.5}$$

as in the theory of univariate orthogonal polynomials of hypergeometric type [25, Chapter 1], where σ and τ are polynomials of at most degree two and one, respectively. Our three distributions solve (4.1.5) in these three cases:

$$\begin{aligned}
\frac{\varrho_N'(x)}{\varrho_N(x)} &= -2x, \quad \sigma(x) = 1, \quad \tau(x) = -2x,\\
\frac{\varrho_G'(x;a)}{\varrho_G(x;a)} &= \frac{a-x}{x}, \quad \sigma(x) = x, \quad \tau(x) = a+1-x,\\
\frac{\varrho_B'(x;a,b)}{\varrho_B(x;a,b)} &= \frac{a-(a+b)x}{x(1-x)}, \; \sigma(x) = x(1-x), \quad \tau(x) = 1+a-(a+b+2)x.
\end{aligned}$$

There also exist limit transitions between these three distributions [25] such as between the beta and gamma,

$$\lim_{b\to\infty} \frac{1}{b}\varrho_B\left(\frac{x}{b};a,b\right) = \varrho_G(x;a), \tag{4.1.6}$$

and between the gamma and normal

$$\lim_{a\to\infty} \sqrt{a}\varrho_G(a + x\sqrt{2a};a) = \varrho_N(x). \tag{4.1.7}$$

Let us now define the Fractional Pearson equation [2]

$$A(x^\alpha)D^\alpha \varrho(x) = B(x^\alpha)\varrho(x), \qquad 0<\alpha<1, \tag{4.1.8}$$

where $A(x)$ is a polynomial of at most degree two and $B(x)$ is a polynomial of degree one. Notice that (4.1.8) converges formally to (4.1.1) as $\alpha \to 1^-$, since ${}^cD^\alpha f(t)$ tends formally to $f'(t)$ as $\alpha \to 1^-$.

In this chapter, we are first interested in solving (4.1.8), for some specific A and B. These solutions provide fractional analogues of beta, gamma and normal distributions (section 4.2). We shall also find appropriate limit transitions between these three distributions (section 4.3) and finally we introduce weighted quasi-orthogonal polynomials (section 4.4).

4.2 Fractional Pearson equations

In this section we solve some fractional differential equations such that their solutions generalize classical Pearson equations for beta, gamma and normal distributions.

4.2.1 Fractional normal density. Let us consider (4.1.8) with $A(x)=1$ and $B(x)=-2x$ as follows:

$${}^cD^\alpha \varrho(x;\alpha) = -2x^\alpha \varrho(x;\alpha), \qquad 0<\alpha<1, \quad x>0. \tag{4.2.1}$$

We can solve (4.2.1) by considering the formal series expansion

$$\varrho(x;\alpha) = \sum_{n=0}^{\infty} a_n(\alpha)x^{n\alpha}, \quad x>0. \tag{4.2.2}$$

So the left hand side of (4.2.1) becomes

$${}^cD^\alpha \varrho(x;\alpha) = \sum_{n=0}^{\infty} a_{n+1}(\alpha)x^{n\alpha}\frac{\Gamma(1+\alpha(n+1))}{\Gamma(1+n\alpha)}. \tag{4.2.3}$$

By replacing (4.2.2) and (4.2.3) in (4.2.1), and by identification, we obtain the following recurrence relation for the coefficients $a_n(\alpha)$,

$$a_n(\alpha) = -2a_{n-2}(\alpha)\frac{\Gamma(\alpha(n-1)+1)}{\Gamma(\alpha n+1)}. \tag{4.2.4}$$

By considering the initial conditions $a_{-1}=0$ and $a_0=1$ we create the fractional normal density defined by

$$\varrho(x;\alpha) = \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2}\,(-2)^{n/2}\,x^{n\alpha}\frac{\prod_{j=0}^{n/2-1}\Gamma(1+\alpha(1+2j))}{\prod_{j=0}^{n/2}\Gamma(1+2\alpha j)}, \quad x>0, \tag{4.2.5}$$

and for $x<0$, we exploit the symmetry $\varrho(-x;\alpha)=\varrho(x;\alpha)$.

When $\alpha\to 1^-$, (4.2.4) becomes

$$a_n = -2\frac{a_{n-2}}{n}.$$

By considering the initial conditions $a_0=1$ and $a_1=0$, (4.2.4) is the recurrence relation for the coefficients a_n in the expansion

$$\varrho_N(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where $\varrho_N(x)$ is defined in (4.1.2). It is also good to know that for the initial conditions $a_0 = 0$ and $a_1 = 1$, (4.2.4) is the recurrence relation for another solution giving rise to $g(x) = \frac{1}{2}\sqrt{\pi}e^{-x^2}\text{erfi}(x)$. Similarly, in the fractional case the recurrence relation (4.2.4) could generate another solution.

We can see in Figure 4.1 that the fractional normal density approximates the classical normal density with α close to 1.

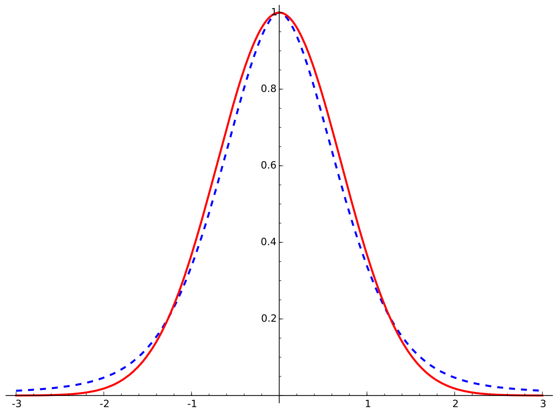

Figure 4.1: Graph of the partial sum defined from (4.2.5) and expanded to $x < 0$ by symmetry with 100 terms and $\alpha = 9/10$ (dashed blue) and $\exp(-x^2)$ (solid red) for $x \in [-3, 3]$.

4.2.2 Fractional gamma density. In this section, let $a > 0$. When we consider (4.1.8) with $A(x) = x$ and $B(x) = \frac{\Gamma(1+a)}{\Gamma(1+a-\alpha)} - x$, we have

$$x^{\alpha}\,{}^{c}D^{\alpha}\varrho(x;a;\alpha) = \left(\frac{\Gamma(1+a)}{\Gamma(1+a-\alpha)} - x^{\alpha}\right)\varrho(x;a;\alpha), \quad x > 0. \tag{4.2.6}$$

The fractional differential equation (4.2.6) can be solved by considering the formal series expansion

$$\varrho(x;a;\alpha) = \sum_{n=0}^{\infty} b_n(a;\alpha)x^{n\alpha+a}. \tag{4.2.7}$$

So the left hand side of (4.2.6) becomes

$$x^{\alpha}\,{}^{c}D^{\alpha}\varrho(x;a;\alpha) = \sum_{n=0}^{\infty} b_n(a;\alpha)\frac{\Gamma(a+n\alpha+1)}{\Gamma(a+(n-1)\alpha+1)}x^{n\alpha+a}. \tag{4.2.8}$$

By replacing (4.2.7) and (4.2.8) in (4.2.6), and by identification, we obtain the following recurrence relation for the coefficients $b_n(a;\alpha)$,

$$b_n(a;\alpha) = \frac{b_{n-1}(a;\alpha)}{\frac{\Gamma(a+1)}{\Gamma(a-\alpha+1)} - \frac{\Gamma(a+n\alpha+1)}{\Gamma(a+(n-1)\alpha+1)}}, \tag{4.2.9}$$

which can be written explicitly as

$$b_n(a;\alpha) = \frac{b_0(a;\alpha)}{\prod_{j=1}^{n} \frac{\Gamma(1+a)}{\Gamma(1+a-\alpha)} - \frac{\Gamma(1+a+j\alpha)}{\Gamma(1+a+(j-1)\alpha)}}.$$

Hence, we define the fractional gamma density by

$$\varrho(x;a;\alpha) = \sum_{n=0}^{\infty} \frac{b_0(a;\alpha)}{\prod_{j=1}^{n} \frac{\Gamma(1+a)}{\Gamma(1+a-\alpha)} - \frac{\Gamma(1+a+j\alpha)}{\Gamma(1+a+(j-1)\alpha)}} x^{n\alpha+a}. \tag{4.2.10}$$

When $\alpha \to 1^-$, (4.2.9) becomes

$$b_n(a) = -\frac{b_{n-1}(a)}{n},$$

which is the recurrence relation for the coefficients b_n in the expansion

$$\varrho_G(x;a) = \sum_{n=0}^{\infty} b_n x^{n+a}$$

where $\varrho_G(x;a)$ is defined in (4.1.3). Notice that as $\alpha \to 1^-$ the fractional differential equation (4.2.6) converges to

$$xy'(x) = (a-x)y(x)$$

which is the differential equation satisfied by $\varrho_G(x;a)$.

We can see in Figure 4.2 that the fractional gamma density approximates the classical gamma density with α close to 1.

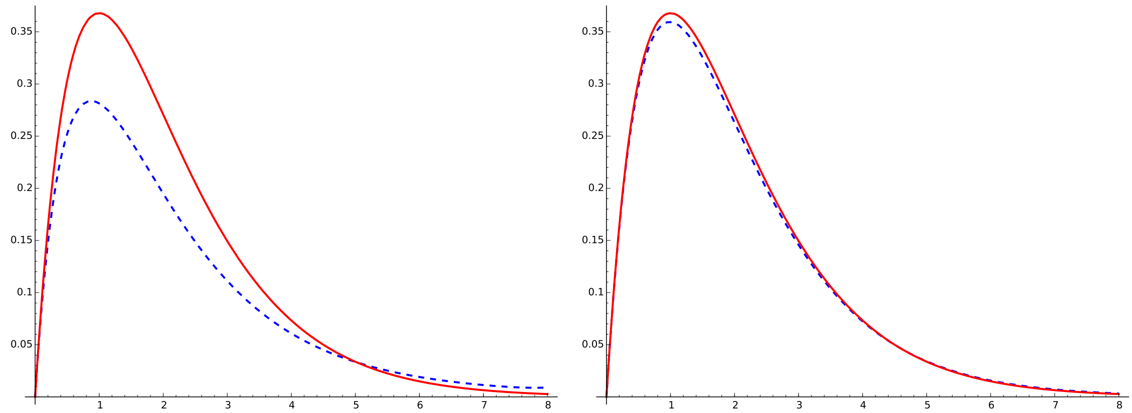

Figure 4.2: Graphs of the partial sum defined in (4.2.10) with 30 terms (dashed blue) and $x^a \exp(-x)$ (solid red) for $x \in [0,8]$, $a = 1$, and $b_0(a;\alpha) = 1$. On the left hand side $\alpha = 9/10$ and on the right hand side $\alpha = 99/100$.

4.2.3 Fractional beta density. In this section, let $a, b > 0$. When we consider (4.1.8) with $A(x) = x(1-x)$ and $B(x) = \frac{\Gamma(a+1)}{\Gamma(a-\alpha+1)} - \frac{\Gamma(a+b+1)}{\Gamma(a-\alpha+b+1)} x^\alpha$, we have

$$x^\alpha (1-x^\alpha)\ {}^{c}D^\alpha \varrho(x;a,b;\alpha) = \left(\frac{\Gamma(a+1)}{\Gamma(a-\alpha+1)} - \left(\frac{\Gamma(a+b+1)}{\Gamma(a-\alpha+b+1)} \right) x^\alpha \right) \varrho(x;a,b;\alpha), \tag{4.2.11}$$

for $x > 0$. We can solve the fractional differential equation (4.2.11) by considering the formal series expansion

$$\varrho(x;a,b;\alpha) = \sum_{n=0}^{\infty} c_n(a,b;\alpha) x^{n\alpha+a}. \tag{4.2.12}$$

So the left hand side of (4.2.11) becomes

$$\begin{aligned} x^\alpha(1-x^\alpha)\ {}^cD^\alpha\varrho(x;a,b;\alpha) \\ &= \sum_{n=0}^{\infty} c_n(a,b;\alpha)\frac{\Gamma(a+n\alpha+1)}{\Gamma(a+(n-1)\alpha+1)}x^{n\alpha+a} \\ &\quad - \sum_{n=1}^{\infty} c_{n-1}(a,b;\alpha)\frac{\Gamma(a+(n-1)\alpha+1)}{\Gamma(a+(n-2)\alpha+1)}x^{n\alpha+a}. \end{aligned} \tag{4.2.13}$$

By replacing (4.2.12) and (4.2.13) in (4.2.11), and by identification, we obtain the following recurrence relation for the coefficients $c_n(a,b;\alpha)$,

$$\begin{aligned} c_n(a,b;\alpha) = c_{n-1}(a,b;\alpha)&\frac{\Gamma(a-\alpha+1)\Gamma(a+\alpha(n-1)+1)}{\Gamma(a-\alpha+b+1)\Gamma(a+\alpha(n-2)+1)} \\ &\times \frac{\Gamma(a+b+1)\Gamma(a+\alpha(n-2)+1)-\Gamma(a-\alpha+b+1)\Gamma(a+\alpha(n-1)+1)}{\Gamma(a+1)\Gamma(a+\alpha(n-1)+1)-\Gamma(a-\alpha+1)\Gamma(a+\alpha n+1)}, \end{aligned} \tag{4.2.14}$$

Hence we give rise to the fractional beta density.

When $\alpha \to 1^-$, (4.2.14) becomes

$$c_n(a,b) = \frac{1}{n}(n-b-1)c_{n-1}(a,b),$$

which is the recurrence relation for the coefficients $c_n(a,b)$ in the expansion

$$x^a(1-x)^b = \sum_{n=0}^{\infty} c_n(a,b)x^n.$$

Notice that as $\alpha \to 1^-$ the fractional differential equation (4.2.11) converges to

$$x(1-x)y'(x) = (a-(a+b)x)y(x)$$

which is the differential equation satisfied by $\varrho_B(x;a,b)$ defined in (4.1.4).

We can see in Figure 4.3 that the fractional beta density approximates the classical beta density with α close to 1.

4.3 Limit transitions

Now, we want to find limit transitions between our fractional normal, gamma and beta densities.

4.3.1 Limit between the fractional beta and gamma densities. Since (4.2.12), then

$$b^a\varrho\left(\frac{x}{b};a,b;\alpha\right) = \sum_{n=0}^{\infty}\frac{1}{b^{n\alpha}}c_n(a,b;\alpha)x^{n\alpha+a}.$$

By using (4.2.14) and (4.2.9), we obtain

$$\lim_{b\to\infty}\frac{1}{b^{n\alpha}}c_n(a,b;\alpha) = b_n(a;\alpha).$$

So

$$\lim_{b\to\infty} b^a\varrho\left(\frac{x}{b};a,b;\alpha\right) = \varrho(x;a;\alpha),$$

which is a generalization of (4.1.6) to the fractional case.

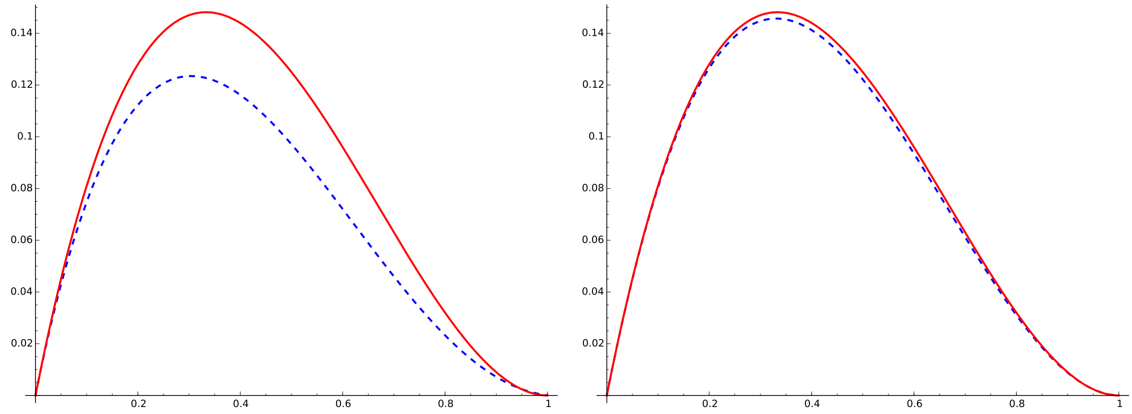

Figure 4.3: Graphs of the partial sum defined in (4.2.12) with 30 terms (dashed blue) and $x^a(1-x)^b$ (solid red) for $x \in [0,1]$ and $a=1$, $b=2$. On the left hand side $\alpha = 9/10$ and on the right hand side $\alpha = 99/100$.

4.3.2 Limit between the fractional gamma and normal densities. To know how to compute the limit between the fractional gamma density and the fractional normal density, we have first to know how to compute the limit between the gamma density and the normal density. Let us define a modification of the gamma density

$$\tilde{\varrho}(x;a) = \varrho_G(a+\sqrt{2a}x;a) = \frac{(a+\sqrt{2a}x)^a \exp(-a-\sqrt{2a}x)}{\Gamma(a+1)}, \tag{4.3.1}$$

which comes from the limit relation (4.1.7). The power series expansion of (4.3.1) gives

$$\tilde{\varrho}(x;a) = \sum_{n=0}^{\infty} \tilde{b}_n(a) x^n, \tag{4.3.2}$$

where

$$\tilde{b}_n(a) = \binom{a}{n} \frac{2^{n/2}\, a^{a-n/2}\, \exp(-a)}{\Gamma(a+1)}\, {}_1F_1\left(\begin{array}{c} -n \\ 1+a-n \end{array} \middle| \, a \right). \tag{4.3.3}$$

We have

$$\lim_{a\to\infty} \sqrt{a}\binom{a}{n} \frac{2^{n/2} a^{a-\frac{n}{2}}\, {}_1F_1(-n; a-n+1; a)}{\exp(a)\, \Gamma(a+1)} = \frac{(-1)^{n/2}\left((-1)^n+1\right)}{2\sqrt{2\pi}\left(\frac{n}{2}\right)!}.$$

So

$$\lim_{a\to\infty} \sqrt{a}\, \tilde{b}_n(a) = \begin{cases} 0, & \text{for} \quad n \quad \text{odd} \\ \dfrac{(-1)^n}{\sqrt{2\pi} n!}, & \text{for} \quad n \quad \text{even,} \end{cases}$$

which corresponds to the coefficients in the power series expansion

$$\varrho(x) = \frac{\exp(-x^2)}{\sqrt{2\pi}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{\sqrt{2\pi} n!}.$$

Hence, we have shown (4.1.7). When we compute the first three coefficients $\tilde{b}_0(a), \tilde{b}_1(a)$ and $\tilde{b}_2(a)$, we obtain

$$\tilde{b}_0(a) = \frac{a^a e^{-a}}{\Gamma(a+1)}, \quad \tilde{b}_1(a) = 0, \quad \text{and} \quad \tilde{b}_2(a) = -\frac{a^a e^{-a}}{\Gamma(a+1)}. \tag{4.3.4}$$

Moreover, when we replace (4.3.2) in the Pearson equation

$$\left(\sqrt{a}+\sqrt{2}x\right)\frac{d}{dx}\tilde{\varrho}(x;a) = -2\sqrt{a}x\,\tilde{\varrho}(x;a), \tag{4.3.5}$$

we arrive to

$$\tilde{b}_{n+1}(a) = \frac{2\sqrt{a}}{(n+1)\sqrt{a}}\tilde{b}_{n-1}(a) - \frac{n\sqrt{2}}{(n+1)\sqrt{a}}\tilde{b}_n(a).$$

By using the initial conditions (4.3.4), we find the same recurrence as in (4.3.3). Hence the function $\tilde{\varrho}(x;a)$ is a solution of the Pearson equation (4.3.5).

For the limit transition between the fractional gamma density and the fractional normal density, let us define the fractional differential equation

$$\left(\frac{\Gamma\left(\sqrt{a}+1\right)}{\Gamma\left(-\alpha+\sqrt{a}+1\right)}+\sqrt{2}x^{\alpha}\right)\,{}^{c}D^{\alpha}\tilde{\varrho}(x;a;\alpha) = -2\frac{\Gamma\left(\sqrt{a}+1\right)}{\Gamma\left(-\alpha+\sqrt{a}+1\right)}x^{\alpha}\,\tilde{\varrho}(x;a;\alpha). \tag{4.3.6}$$

Assume that $\tilde{\varrho}(x;a;\alpha)$ can be written as power series expansion

$$\tilde{\varrho}(x;a;\alpha) = \sum_{n=0}^{\infty}\tilde{b}_n(a;\alpha)x^{n\alpha}. \tag{4.3.7}$$

Then by replacing (4.3.7) in (4.3.6), we obtain the recurrence relation

$$\tilde{b}_n(a;\alpha) = -\frac{\Gamma(\alpha(n-1)+1)}{\Gamma(\alpha n+1)}\left(\frac{\sqrt{2}\tilde{b}_{n-1}(a;\alpha)\Gamma\left(-\alpha+\sqrt{a}+1\right)\Gamma(\alpha(n-1)+1)}{\Gamma\left(\sqrt{a}+1\right)\Gamma(\alpha(n-2)+1)}+2\tilde{b}_{n-2}(a;\alpha)\right). \tag{4.3.8}$$

Since we want that $\tilde{b}_n(a;\alpha)$ tends to $\tilde{b}_n(a)$ as $\alpha\to 1^-$, we decide to take the same initial conditions as in (4.3.4). That is,

$$\tilde{b}_0(a;\alpha) = \frac{a^a e^{-a}}{\Gamma(a+1)} \quad \text{and} \quad \tilde{b}_1(a;\alpha) = 0. \tag{4.3.9}$$

For $n=2$ and by replacing (4.3.9) in (4.3.8), we obtain $\tilde{b}_2(a;\alpha) = -\frac{2a^a e^{-a}\Gamma(\alpha+1)}{\Gamma(a+1)\Gamma(2\alpha+1)}$ which tends indeed to $\tilde{b}_2(a)$ as $\alpha\to 1^-$. We can then prove easily by induction that

$$\lim_{\alpha\to 1^-}\tilde{b}_n(a;\alpha) = \tilde{b}_n(a) \quad \text{and} \quad \lim_{a\to\infty}\sqrt{a}\,\tilde{b}_n(a;\alpha) = \frac{1}{\sqrt{2\pi}}a_n(\alpha).$$

Hence, we have found the limit transition between the fractional gamma density and the fractional normal density which is

$$\lim_{a\to\infty}\sqrt{a}\tilde{\varrho}(x;a;\alpha) = \frac{1}{\sqrt{2\pi}}\varrho(x;\alpha).$$

4.4 Quasi-polynomials orthogonal with respect to fractional densities

We define a quasi-polynomial P_n of order $n\in\mathbf{N}$ with commensurate power $\beta>0$ [32] as a polynomial in t^{β} such that

$$P_n(t^{\beta}) = P_{n,\beta}(t) = \sum_{i=0}^{n} a_{n,i}t^{i\beta}.$$

To compute numerically the fractional integral with unit upper integration

$$\frac{1}{\Gamma(\alpha)} \int_0^1 f(\tau)\,(1-\tau)^{\alpha-1}\,d\tau,$$

it has been considered in [32] the concept of orthogonality as

$$\frac{1}{\Gamma(\alpha)} \int_0^1 P_{n,\beta}(\tau;\alpha)\,P_{m,\beta}(\tau;\alpha)\,(1-\tau)^{\alpha-1}\,d\tau = 0, \quad n \neq m.$$

Moreover, in [11] a family of nonstandard Gauss-Jacobi-Lobatto quadratures for numerical evaluation at integrals of the form

$$\int_{-1}^1 f'(x)(1-x)^{\alpha}dx, \quad \alpha > -1$$

has been derived and applied to approximation of the usual fractional derivative. Here, we present an algorithm that generates the sequence of quasi-polynomials orthonormal with respect to the fractional gamma and beta densities $\rho(x;\alpha)$ on their corresponding supports I, extending recent results in [38] to the fractional case. We shall consider $\rho(x;\alpha) = \varrho(x;a;\alpha)$ and $I = [0,+\infty)$ for the gamma case, and $\rho(x;\alpha) = \varrho(x;a,b;\alpha)$ and $I = (0,1)$ for the beta case.

Let $f, g, p \in L_2(I)$, where $L_2(I)$ is the set of square integrable real-valued functions in the Lebesgue sense over the interval I and $p(x) \geq 0$. We define the inner product by

$$\langle f, g\rangle_p = \int_I f(x)g(x)p(x)dx.$$

A sequence of polynomials $\{P_n(x)\}_n$ with deg$P_n = n$, $n = 0, 1, 2, \ldots,$ is said to be orthogonal with respect to p if

$$\langle P_n, P_m\rangle_p = 0, \quad n \neq m. \tag{4.4.1}$$

The following result ensures the existence of the sequence of orthogonal polynomials [4].

4.4.1 Lemma. By using the notation above, for a given weight function $p \in L_2(I)$, $p \geq 0$, there exists a sequence of monic orthogonal polynomials $P_n(x) = x^n + \cdots$ such that (4.4.1) holds true. Moreover, the polynomials $P_n(x)$ can be computed by using the following three term recurrence relation

$$P_0(x) = 1, \quad P_1(x) = x - \frac{\langle x, 1\rangle_p}{\langle 1, 1\rangle_p},$$
$$P_{n+1}(x) = \left(x - \frac{\langle xP_n, P_n\rangle_p}{\langle P_n, P_n\rangle_p}\right) P_n(x) - \frac{\langle P_n, P_n\rangle_p}{\langle P_{n-1}, P_{n-1}\rangle_p} P_{n-1}(x).$$

Let $\rho(x;\alpha)$ be the fractional gamma or the fractional beta density obtained in the previous sections. Since I remains unchanged under the transformation $x = t^\beta$, we have

$$\int_I P_n(t^\beta;\alpha)P_m(t^\beta;\alpha)\rho(t;\alpha)dt = \int_I P_n(x;\alpha)P_m(x;\alpha)\rho(x^{1/\beta};\alpha)\frac{x^{(1-\beta)/\beta}}{\beta}dx.$$

Let us consider

$$p(x) = \varrho(x^{1/\beta};\alpha)\frac{x^{(1-\beta)/\beta}}{\beta}.$$

We have $p(x) > 0$ on I and for all $n \in \mathbf{N}$, $\int_I p(x)x^n dx \in \mathbf{R}$. So p is a weight function. Thus, Lemma 4.4.1 ensure that there exists a sequence of monic orthogonal polynomials $P_n(x,\alpha)$ such that (4.4.1) holds true. Then, there exists a sequence of quasi-orthogonal polynomials $P_{n,\beta}(x,\alpha)$ with respect to $\rho(x;\alpha)$.

As a consequence of some classical results in [4], we have the following lemma.

4.4.2 Lemma. Let $\varrho(x;\alpha)$ be the fractional gamma or the fractional beta densities obtained in the previous sections, and $\{P_n(t^\beta;\alpha)\}$ the sequence of quasi-polynomials orthogonal with respect to $\varrho(x;\alpha)$. Each orthogonal quasi-polynomial $P_n(t^\beta;\alpha)$ has exactly n real and distinct roots, all of them in I.

Let us define

$$\mathbf{X}_n(\beta) := (1, x, x^{2\beta}, \dots, x^{n\beta})^T, \quad n \geq 0,$$

and

$$m_n(\alpha,\beta) := \int_I x^{n\beta}\varrho(x;\alpha)dx, \quad n \geq 0.$$

Let us also introduce the following matrix

$$\mathbf{M}_n(\alpha,\beta) := \int_I \mathbf{X}_n(\beta)(\mathbf{X}_n(\beta))^T\varrho(x;\alpha)dx.$$

of size $(n+1)\times(n+1)$, where the integral is element by element. Here, $\mathbf{M}_n(\alpha,\beta)$ is a symmetric and positive-definite matrix [38]. Let

$$\mathbf{M}_n(\alpha,\beta) = \mathbf{L}_n(\alpha,\beta)(\mathbf{L}_n(\alpha,\beta))^T,$$

be its Cholesky decomposition [12], where $\mathbf{L}_n(\alpha,\beta)$ is a lower triangular matrix with nonnegative diagonal entries. Let us also define

$$\mathbf{T}_n(\alpha,\beta) := (\mathbf{L}_n(\alpha,\beta))^{-1}, \tag{4.4.2}$$

and the vector of $n+1$ quasi-polynomials

$$\mathbf{P}_n(\alpha,\beta) := \mathbf{T}_n(\alpha,\beta)\,\mathbf{X}_n(\beta). \tag{4.4.3}$$

Then, we have the following theorem.

4.4.3 Theorem. *The sequence $\{\mathbf{P}_n(\alpha,\beta)\}_{n\geq 0}$ defined in (4.4.3) satisfies the orthonormality relation*

$$\int_I \mathbf{P}_n(\alpha,\beta)(\mathbf{P}_n(\alpha,\beta))^T\varrho(x;\alpha)dx = \mathbf{I}_{n+1}, \tag{4.4.4}$$

where $\mathbf{I}_n$ denotes the identity matrix of size n.

Proof. Since

$$\begin{aligned}
\int_I \mathbf{P}_n(\alpha,\beta)(\mathbf{P}_n(\alpha,\beta))^T\varrho(x;\alpha)dx &= \int_I \mathbf{T}_n(\alpha,\beta)\,\mathbf{X}_n(\beta)(\mathbf{X}_n(\beta))^T\,(\mathbf{T}_n(\alpha,\beta))^T\varrho(x;\alpha)dx \\
&= \mathbf{T}_n(\alpha,\beta)\left(\int_I \mathbf{X}_n(\beta)(\mathbf{X}_n(\beta))^T\,\varrho(x;\alpha)dx\right)(\mathbf{T}_n(\alpha,\beta))^T \\
&= \mathbf{T}_n(\alpha,\beta)\mathbf{M}_n(\alpha,\beta)(\mathbf{T}_n(\alpha,\beta))^T \\
&= \mathbf{T}_n(\alpha,\beta)\mathbf{L}_n(\alpha,\beta)(\mathbf{L}_n(\alpha,\beta))^T(\mathbf{T}_n(\alpha,\beta))^T \\
&= \mathbf{T}_n(\alpha,\beta)\mathbf{L}_n(\alpha,\beta)(\mathbf{T}_n(\alpha,\beta)\mathbf{L}_n(\alpha,\beta))^T \\
&= \mathbf{I}_{n+1} \quad \text{by using (4.4.2).}
\end{aligned}$$

□

4.4.4 Numerical examples and conjectures. Let us consider the weight function $\varrho(x; a, b; \alpha)$ defined in (4.2.12) in the particular case $\alpha = 1/2$, $a = 1$, and $b = 2$, normalized as $\int_0^1 \varrho(x; 1, 2; 1/2)dx = 1$, and let us fix $\beta = 1/2$. In this particular case, we can approximate the matrix $\mathbf{M}_3(1, 2; 1/2, 1/2) = \mathbf{M}_3(1/2, 1/2)$ as

$$\mathbf{M}_3(1,2;1/2,1/2) = \begin{pmatrix} 1. & 0.522651 & 0.305996 & 0.194547 \\ 0.522651 & 0.305996 & 0.194547 & 0.131658 \\ 0.305996 & 0.194547 & 0.131658 & 0.0935493 \\ 0.194547 & 0.131658 & 0.0935493 & 0.0691131 \end{pmatrix},$$

with Cholesky factor $\mathbf{L}_3(1, 2; 1/2, 1/2) = \mathbf{L}_3(1/2, 1/2)$

$$\mathbf{L}_3(1,2;1/2,1/2) = \begin{pmatrix} 1. & 0. & 0. & 0. \\ 0.522651 & 0.181198 & 0. & 0. \\ 0.305996 & 0.191053 & 0.0390292 & 0. \\ 0.194547 & 0.165443 & 0.061751 & 0.00893294 \end{pmatrix}.$$

Then, we have that $\mathbf{P}_3(1, 2; 1/2, 1/2) = \mathbf{P}_3(1/2, 1/2)$ given by

$$\mathbf{P}_3(1,2;1/2,1/2) = \begin{pmatrix} 1. \\ 5.51883\sqrt{x} - 2.88442 \\ 25.6218x - 27.0154\sqrt{x} + 6.27942 \\ 111.945x^{3/2} - 177.117x + 84.5384\sqrt{x} - 11.7656 \end{pmatrix}$$

satisfies the orthogonality relation (4.4.4).

Similarly, we have

$$\mathbf{P}_3(1,2;0.4,1/2) = \begin{pmatrix} 1 \\ 5.41862\sqrt{x} - 2.66051 \\ 24.7858x - 25.0858\sqrt{x} + 5.49757 \\ 107.31x^{3/2} - 164.694x + 75.4371\sqrt{x} - 9.92172 \end{pmatrix}$$

and

$$\mathbf{P}_3(1,2;0.6,1/2) = \begin{pmatrix} 1 \\ 5.61306\sqrt{x} - 3.07125 \\ 26.3874x - 28.6843\sqrt{x} + 6.95742 \\ 116.164x^{3/2} - 188.006x + 92.4758\sqrt{x} - 13.4001 \end{pmatrix}.$$

Using the same arguments as in [38] and Theorem 4.4.3, we have that each row of $\mathbf{P}_n(\alpha, \beta)$ has exactly n real and distinct zeros on I.

By using Mathematica [42], we have performed a number of numerical experiments approximating the zeros of the quasi-polynomials

$$P_m(x; \alpha, \beta) = (\mathbf{P}_n(\alpha, \beta))_m,$$

where $n \geq m$. The following conjectures will be considered in future research:

1. The zeros of $P_n(x; \alpha, \beta)$ are increasing functions of α.
2. The zeros of $P_n(t; \alpha, \beta)$ are decreasing functions of β.

3. For the fractional beta density,

   (a) The zeros of $P_n(t;\alpha,\beta) = P_n(t;a,b;\alpha,\beta)$ are increasing functions of a.

   (b) The zeros of $P_n(t;a,b;\alpha,\beta)$ are decreasing functions of b.

4. For the fractional gamma density, the zeros of $P_n(t;\alpha,\beta) = P_n(t;a;\alpha,\beta)$ are increasing functions of a.

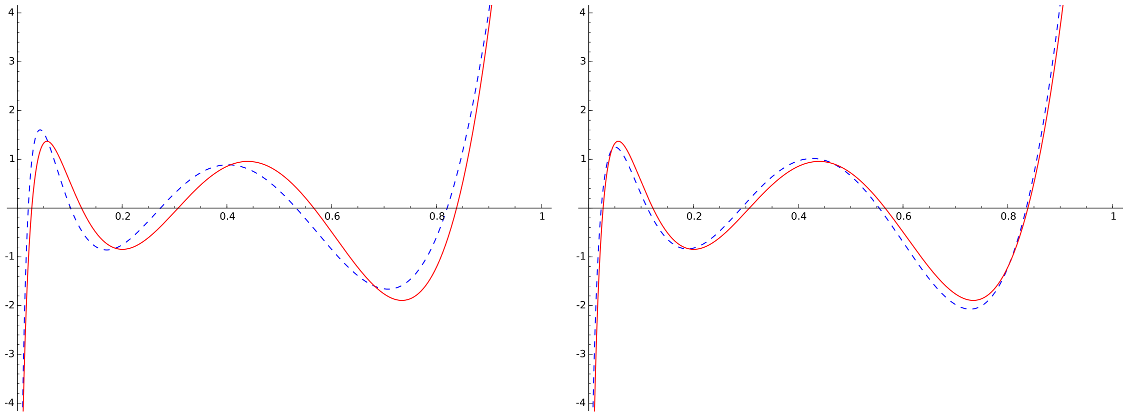

Figure 4.4: Graphs of quasi-polynomials orthogonal with respect to fractional beta distribution with $n = 5$, $a = 1$, and $b = 2$. On the left, for $\alpha = \beta = 0.5$ (dashed blue) and $\alpha = 0.5$, $\beta = 0.6$ (solid red). The roots of these quasi-polynomials are 0.0208402, 0.101603, 0.273096, 0.532213, and 0.820418 for $\alpha = \beta = 0.5$; and 0.0275359, 0.122522, 0.306668, 0.566053, and 0.838286 for $\alpha = 0.5$ and $\beta = 0.6$. On the right, for $\alpha = 0.4$, $\beta = 0.6$ (dashed blue) and $\alpha = 0.5$, $\beta = 0.6$ (solid red). The roots in the case $\alpha = 0.4$ and $\beta = 0.6$ are 0.0239517, 0.112001, 0.291514, 0.55457, and 0.835314.

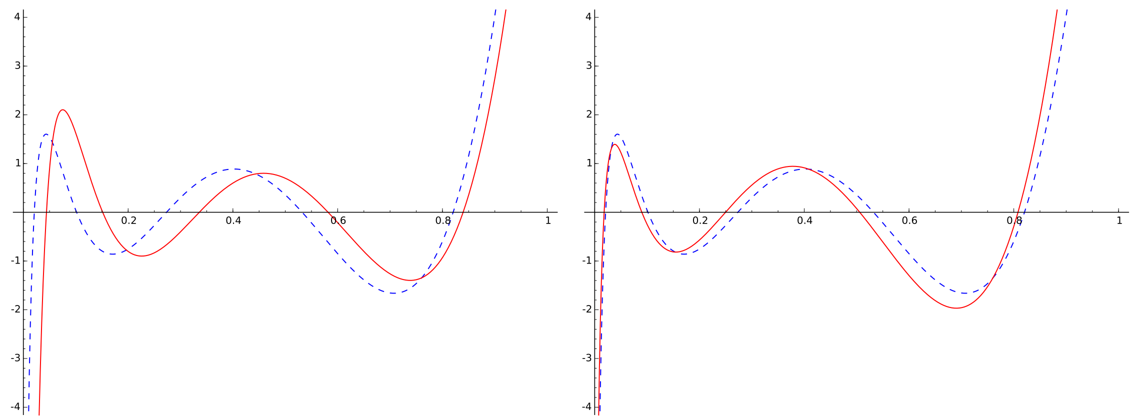

Figure 4.5: Graphs of quasi-polynomials orthogonal with respect to fractional beta distribution with $n = 5$, $\alpha = \beta = 1/2$. On the left, for $a = 1$, $b = 2$ (dashed blue) and $a = 2$, $b = 2$ (solid red). The roots of these quasi-polynomials are 0.0208402, 0.101603, 0.273096, 0.532213, and 0.820418, for $a = 1$, $b = 2$; and 0.0440961, 0.151432, 0.335739, 0.581802, and 0.839372 for $a = b = 2$. On the right, for $a = 1$, $b = 2$ (dashed blue) and $a = 1$, $b = 3$ (solid red). The roots in the case $a = 1$ and $b = 3$ are 0.018028, 0.0899672, 0.249686, 0.505895, 0.808125.

5. Conclusion

Fractional calculus is an extension of classical differential calculus where the order of derivatives is not an integer but a real number. For a long time, its theory developed only as a pure theoretical field of mathematics. However, in the last decades, it was found in many real-world applications.

In this essay, we have introduced fractional calculus by giving some basic definitions and theorems that appear in this field. Then we studied a real-world application of fractional calculus by solving analytically and numerically the fractional diffusion equation with drift, this partial differential equation which appears in some real-world applications related with anomalous super-dispersion. We found analytically that the solution of the fractional diffusion equation with drift cannot be expressed in terms of elementary functions. Numerically, we found that the fractional term of the fractional diffusion equation with drift behaves as a mix of pure transport and pure diffusion, where the quantity of transport and diffusion respectively depends on the derivative order of the fractional term in a non-trivial way.

Furthermore, we found the fractional analogues of the normal, beta and gamma distributions by solving the fractional Pearson equation in certain cases. These analogues are called fractional normal, gamma and beta densities. We established limit transitions between them and then we introduced weighted orthogonal quasi-polynomials with respect to these new fractional densities.

Moreover, some conjectures about the zeros of these orthogonal quasi-polynomials were stated, conjectures which we would like to consider in future research. As another future line of research, convergence analysis for the scheme of numerical solution of the fractional diffusion equation with drift could also be done.

Appendix A. Python code for solving the fractional diffusion equation

```
# diffusion equation with drift

# first we import relevant modules and functions.
from numpy import zeros
from numpy.linalg import solve
from scipy.special import gamma
from numpy import linspace

# We give initial conditions f(x) to initialize C_0

# one initial condition can be the normal density of
# mean mu and standard deviation sigma
def f(x, sigma, mu):
    y = (1.0/(sigma * sqrt(2*pi)))*exp(-((x-mu)/sigma)**2)
    return y/max(y)

#another one can be f(x) = 1 - 1_{0}(x)
def f(x):
    H = zeros([len(x)])
    H[50:] = 1
    return H

#We define our algorithm to solve the fractional diffusion equation.
def Numeric(M, N, E, B, alpha, C0):
# We initialize C as a null matrix of size (M+1)x(N+1)
    C = zeros([M+1, N+1])

# We initialize its first column
    C[:,0] = C0

# we initialize A as a null matrix of size (M+1)x(M+1)
    A = zeros([M+1, M+1])

# we initialize g as a null vector of size (M+1)
    g = zeros([M+1])

# since g(0) = 1 and g(1) = -\alpha, we replace it also in g.
    g[0] = 1
    g[1] = - alpha

# So for the rest of value of the vector g, we replace it by using the
# formula in the essay.
    for k in range(2,M+1):
```

```
# in python, for k >= 171, \gamma(k) =  \infinity.
        if k<170:
            g[k] = gamma(k - alpha)/ (gamma(- alpha)*gamma(k+1)*1.0)
        else:
            g[k] = 0
    # end for

# now we fill the matrix A as follow:
    for i in range(1,M):
        if alpha == 1:
            A[i,i   ] = 1 + E - B
            A[i,i-1 ] = -E + B
        elif alpha == 2:
            A[i,i+1 ] = -B
            A[i,i   ] = 1 + E + 2*B
            A[i,i-1 ] = -E - B
        else:
            A[i,i+1 ] = -g[0]*B
            A[i,i   ] = 1 + E - g[1]*B
            A[i,i-1 ] = -E - g[2]*B
            for j in range(1,i-1):
                A[i,j] = -g[i-j+1]*B
            # end for
        #end if
    # end for
    A[0,0] = 1
    A[0,1:] = 0
    A[M,:M] = 0
    A[1:,0] = 0
    A[:M-1,M] = 0
    A[M,M-1] = -1
    A[M,M] = 1

# We fill the matrix C
# since we want C_{M}^{n+1} = C_{M-1}^{n+1}, then for each value of j, we
# keep C^j in C then we create a new vector CC which is the vector C^j in
# which we replace the last position by $0$. That vector CC helps us to
# compute C^{j+1} by solving A C^{j+1} = CC
    for j in range(1,N):
        CC = zeros([M+1])
        CC[: -1] = C[:-1,j-1]
        C[:,j] = solve(A,CC)
    # end for
    return(C)

if __name__=='__main__':
    M = 1000  #Number of division in space
```

```
N = int(2.5*M)   #Number of division in time
plotFontsize = 15.5
L = 0.0
R   = 1.0
T   = 1.0
v   = 0.5
D   = 0.1
mu = 0.3
sigma = 0.05
X0  = linspace(L, R, M+1)
T0  = linspace(0, T, N+1)
h   = (R - L)/M*1.0
tau = T/N*1.0
vectalpha = linspace(1, 2, 5)
for i in range(len(vectalpha)):
    alpha = vectalpha[i]
    E   = v*1.0*tau/h
    B   = D*1.0*tau/(h**alpha)
    #C0 = f(X0,sigma, mu)
    C0 = f(X0)
    C = Numeric(M, N, E, B, alpha, C0)
    clf()
    L0 = plot(X0,C[:,0])
    L1 = plot(X0,C[:,0.2*N])
    L2 = plot(X0,C[:,0.4*N])
    L3 = plot(X0,C[:,0.6*N])
    legend(('t = 0','t = 0.2','t = 0.4','t = 0.6'),fontsize=plotFontsize,
   loc = 'upper right')
    title(r'alpha = %0.3f'%(alpha), fontsize=plotFontsize)
    xlabel('Spatial coordinate, x',fontsize=plotFontsize)
    ylabel('Concentration, C(x,t)',fontsize=plotFontsize)
    setp(gca().get_xticklabels(),fontsize=plotFontsize)
    setp(gca().get_yticklabels(),fontsize=plotFontsize)
    savefig('.../Bdiffusion_drift_alpha-%0.3f.pdf' %(alpha))
```

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