Existence and Uniqueness of Viscosity Solutions (for Initial Value Problems) of Hamilton-Jacobi Equations and Optimal control Theory

Woldegebriel Assefa WOLDEGERIMA (assefa@aims-cameroon.org)
African Institute for Mathematical Sciences (AIMS)
Cameroon

Supervised by: Dr. Raoul Ayissi
University of Yaounde I, Cameroon

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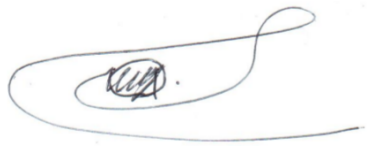
Abstract

In this essay, we present the mathematical theory of viscosity solutions for initial value problems (IVP) of Hamilton-Jacobi equations (HJE). The focus is on the existence, consistency and uniqueness of viscosity solutions. We carry out a study of the connection between terminal value problems (TVP) for Hamilton-Jacobi equations called, Hamilton-Jacobi-Bellman equation (HJBE), and optimal control theory. It is also proved that a value function for optimal control problem provides a unique viscosity solution of the TVP for HJBE. Eventually, we introduced a Hopf-Lax formula which provides a unique viscosity solution. In most of the references used, the theorems and lemmas related to existence, consistency and uniqueness of viscosity solutions are proved using local maximum and left for the reader to prove using local minimum. In this essay, an effort is made to use local minimum.

Keywords: Hamilton-Jacobi Equation (HJE), viscosity solution, optimal control, value function, Hamilton-Jacobi-Bellman equation (HJBE), Hopf-Lax formula.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



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Contents

Abstract **i**

1 Introduction, Background and Preliminaries **1**

1.1 The main concern of the essay 1

1.2 Background from Mechanics 2

1.3 Preliminaries 2

2 Viscosity Solutions **5**

2.1 Why viscosity solutions 5

2.2 Consistency of viscosity solutions 7

2.3 Uniqueness of viscosity solutions 11

3 Optimal control theory **17**

3.1 The concept 17

3.2 Optimal control theory and its connection to HJE 18

3.3 Terminal Value Problem for Hamilton-Jacobi-Bellman Equation 21

4 The Hopf-Lax Formula **25**

4.1 Motivation for Hopf-Lax formula 25

4.2 The Hopf-Lax formula as viscosity solution 28

5 Conclusion **33**

References **35**

1. Introduction, Background and Preliminaries

This chapter deals with general introduction, background and some basic concepts and definitions as preliminaries that we will use in this essay.

1.1 The main concern of the essay

First order non-linear partial differential equations are one of the most important equations in the application of partial differential equations (PDE) in Science and Engineering. Solving (or developing methods of solving for) these equations was not an easy task for the past many years although some methods such as the method of characteristics and the method of variational principle has been developed. The solutions obtained by the method of characteristics are not smooth enough in a given global domain so that they fail to describe the required results of the PDE in the global domain. We may not be guaranteed with the Existence, uniqueness, consistency, stability in the topology of uniform convergence, continuity and other properties. The failure of weak solutions to describe physical situations every where in a global domain, and the need of the application non-linear partial differential equations in Science, economics and engineering to have global solutions has became an interesting issue for many Mathematicians and engineers for the past many years. As a consequence of this, the notion of Viscosity solution has been introduced by the two mathematicians M. G. Crandall and Pierre-Lious Lions. Our focus is on the mathematical theory of viscosity solutions for special type of non-linear partial differential equations, called Hamilton-Jacobi Equations (HJEs).

The Hamilton-Jacobi equations are a special type of non-linear first order PDEs whose global results are possible, and thus many mathematicians studied the viscosity solutions of HJEs since they provide global solutions [B$^+$83]. In this essay, we focus on the mathematical theory of viscosity solutions for first order Hamilton-Jacobi equation (HJE) of the form $u_t(t,x) + H(x, \nabla u(t,x)) = 0$, with an emphasis on viscosity solutions of initial value problems for these Hamilton-Jacobi equations. For Viscosity solutions the general existence, continuity, uniqueness, stability with respect to the uniform convergence norm and other properties holds true, unlike the solutions obtained by the method of characteristics[CEL84], [B$^+$83].

The essay has two main parts: the notion of viscosity solutions and optimal control theory.

The second chapter deals with the notion of viscosity solutions for an initial value problem of a Hamilton-Jacobi equation, which is an appropriate and useful notion of weak solutions. This chapter answers some basic questions in the solution of partial differential equations (PDEs), such as the existence, consistency and uniqueness of solutions.
In chapter 3, we introduce optimal control theory and its relation to HJE. Then we introduce a terminal value problem for HJE called Hamilton-Jacobi-Bellman equation (HJBE), which is a type of HJE where the Hamiltonian H is a minimum function.The chapter ends with a uniqueness theorem that provides one way of obtaining viscosity solutions: the Value function. The fourth chapter covers Hopf-Lax formula for viscosity solutions, which is another approach for obtaining unique viscosity solution.

1.2 Background from Mechanics

This background of Hamilton-Jacobi theory is adapted from the books of Aliyu [Ali11] and Arnold [Arn89].

Hamilton-Jacobi theory [Ali11], [Arn89] is an extension of Lagrangian mechanics and Calculus of variations which deals with a Hamiltonian system of equations of motion with the task of finding extreme values by reducing the problem into the integration (solving) of a first order PDE, called Hamilton-Jacobi equation. The Hamilton-Jacobi theory was developed by Hamilton for problems in wave optics and geometrical optics and later he extended his idea to dynamics. Jacobi applied the concepts of Hamilton to Calculus of variations. He suggested that a function of the form $S(t,x) := \int_r L dt, \quad x \in \mathbb{R}^n$ should be used in solving problem of classical calculus of variations, where r is a curve connecting points (t_0, x_0) and (t, x) and L is the Lagrangian function of the mechanical system.

The HJE was named after an Irish Mathematician, astronomer and physicist William Rowan Hamilton (1805-1865) and a German Mathematician Carl Gustav Jacobi (1804-1851) [Ali11], [CL83]. The HJE is known as a PDE which arises in the study of Lagrangian or Hamiltonian Mechanics and their relation to Calculus of Variations in studying control theory. It has become a target for many mathematicians who study global solutions [B+83].

From the calculus of variations it has been discovered that the variational approach to the problems of mechanics could be used to solve problems of optimal controls [Ali11]. A Hamilton-Jacobi equation that governs the behaviour of mechanical systems also governs the behaviour of an optimally controlled system, which is one of the importance of HJE in its connection to optimal control theory. An American Mathematician, Richard Bellman (1920-1984),[Ali11] has developed the discrete-time equivalence of a HJE which is called dynamic programming principle and a combined work of these great mathematician introduced a partial differential equation known as Hamilton-Jacobi -Bellman equation (HJBE) which is the back bone of optimal control theory.

1.3 Preliminaries

The following basic concepts and ideas are taken from the book [Eva10, chapter 3] as preliminaries.

We consider $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a given smooth function, called the Lagrangian, which is the difference of kinetic energy and potential energy of a system in mechanics and $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the Hamiltonian, which is the total energy of a system in mechanics. We take $L = L(x, q)$ and $H = H(x, p)$, for $x,\ q,\ p \in \mathbb{R}^n$.

Consider two fixed points $x, y \in \mathbb{R}^n$ and a family of curves in C^2 with starting point y at time $t = t_0$ and reaches the point x at time $t > 0$, for $0 \leq t_0 < t$, of the form

$$\mathcal{U} = \left\{ \mathbf{w}(\cdot) \in C^2\left([t_0, t]\right) : \mathbf{w}(t_0) = y,\ \mathbf{w}(t) = x \right\}. \tag{1.3.1}$$

One of the problems in the study of calculus of variations and mechanics is to minimize an action functional of the type

$$S[\mathbf{w}(\cdot)] = \int_{t_0}^{t} L(\mathbf{w}(s), \dot{\mathbf{w}}(s)) ds, \qquad t_0 < s < t. \tag{1.3.2}$$

We seek a curve $\mathbf{x}(\cdot) \in \mathcal{U}$ such that

$$S[\mathbf{x}(\cdot)] = \min_{\mathbf{w}(\cdot)\in\mathcal{U}} S[\mathbf{w}(\cdot)], \quad \text{i.e.} \quad \int_{t_0}^{t} L\left(\mathbf{x}(s), \dot{\mathbf{x}}(s)\right) ds = \min_{\mathbf{w}(\cdot)\in\mathcal{U}} \int_{t_0}^{t} L(\mathbf{w}(s), \dot{\mathbf{w}}(s)) ds. \tag{1.3.3}$$

The Hamilton's least action principle is a variational principle that, when applied to the action of a mechanical system, can be used to obtain the equations of motion for that system. It states that a trajectory $\mathbf{x}(\cdot)$ of a mechanical system coincides with extremal of the action functional (1.3.2). That is, a path $\mathbf{x}(\cdot) \in \mathcal{U}$ in configuration space between fixed states y at time t_0 and x at time $t > 0$ minimizes the action functional (1.3.2).

We here after assume that there exists such a function $\mathbf{x}(\cdot) \in \mathcal{U}$ and consider that $''x''$ as the variable we substitute for $\mathbf{w}(s)$ and $''q''$ as the variable we substitute for $\dot{\mathbf{w}}(s)$.

Let's set $\mathbf{p}(\cdot) := \nabla_q L(\mathbf{x}(\cdot), \dot{\mathbf{x}}(\cdot))$, where $\mathbf{p}(\cdot)$ is called generalized momentum corresponding to the position $\mathbf{x}(\cdot)$ and velocity $\dot{\mathbf{x}}(\cdot)$.

Suppose that the equation $p = \nabla_q L(x, q)$ can be uniquely solved for q as a function of x and p, that is, $q = \mathbf{q}(x, p)$. We denote $\nabla_x L = \left(\frac{\partial L}{\partial x_1}, \frac{\partial L}{\partial x_2}, \cdots, \frac{\partial L}{\partial x_n}\right)$ and $\nabla_q L = \left(\frac{\partial L}{\partial q_1}, \frac{\partial L}{\partial q_2}, \cdots, \frac{\partial L}{\partial q_n}\right)$.

Then we state the following definitions and concepts. Note that the notations we are using are different from the ones familiar in Mechanics. But, our notations are better in the study of PDE theory.

1.3.1 Definition. a) A system of Euler-Lagrange equations is given by

$$-\frac{d}{ds}\left(\nabla_q L(\mathbf{x}(s), \dot{\mathbf{x}}(s)) + \nabla_x L(\mathbf{x}(s), \dot{\mathbf{x}}(s)\right) = 0. \tag{1.3.4}$$

b) The **Hamiltonian** H associated with the Lagrangian L is defined by

$$H(x, p) := p \cdot \mathbf{q}(x, p) - L(x, \mathbf{q}(x, p)). \tag{1.3.5}$$

That is, $H(x, p) := p \cdot q - L(x, q), \quad x, p \in \mathbb{R}^n$ and $q = \mathbf{q}(x, p)$.

c) Let the mapping $q \longmapsto L(x, q)$ be convex in q and $\lim_{|q|\to\infty} \frac{L(x,q)}{|q|} = \infty$. Then the **Legendre transform** of the Lagrangian L, denoted by L^*, is

$$L^*(x, p) = \sup_{q\in\mathbb{R}^n} \{p.q - L(x, q)\}, \quad (x,\ p \in \mathbb{R}^n). \tag{1.3.6}$$

1.3.2 Remark. Let the mapping $q \longmapsto L(x, q)$ is convex in q and $\lim_{|q|\to\infty} \frac{L(x,q)}{|q|} = \infty$. Then

i) $\sup_{q\in\mathbb{R}^n}\{p \cdot q - L(x, q)\} < \infty$.

ii) the "sup" in (1.3.6) is a "max" and $\exists\, q^* \in \mathbb{R}^n$ such that $\sup_{q\in\mathbb{R}^n}\{p.q - L(x, q)\} = p.q^* - L(x, q^*)$.

iii) $L^*(x, p) = p.q - L(x, q) = H(x, p)$.

That is, we have convex duality for H and L : $L^* = H, \quad H^* = L, \quad L^{**} = L$.

Proof. See [Eva10, page 122] □

The notion of viscosity solutions and HJE have great applications in different areas of mathematics, financial mathematics, engineering and computer science; such as, equations arising in optimal control theory, stochastic calculus, differential games, geometric equations (for example mean curvature, distance function), surface evolutions, eikonal equations, image processing, and $3D$ reconstructions [CP+11].

1.3.3 Remark. i) (Jensen's inequality). [Eva10, page-621] Assume $f : \mathbb{R} \to \mathbb{R}$ is convex and $\Omega \subseteq \mathbb{R}^n$ is open and bounded. Let $u : \Omega \to \mathbb{R}$ be summable. Then

$$f\left(\frac{1}{|u|}\int_\Omega u dx\right) \leq \frac{1}{|f(u)|}\int_\Omega f(u)dx. \tag{1.3.7}$$

ii) (Gronwall's inequality, differential form). [Eva10, page-624] Let $f(\cdot)$ be a nonnegative, absolutely continuous on $[0,T]$ such that $f'(t) \leq g(t)f(t) + h(t)$, for a.e t in $[0,T]$, where $g(t)$ and $h(t)$ are nonnegative and summable on $[0,T]$. Then

$$f(t) \leq \exp\left(\int_0^t g(s)ds\right)\left[f(0) + \int_0^t h(s)ds\right], \quad \text{for} \quad 0 \leq t \leq T. \tag{1.3.8}$$

iii) (Gronwall's inequality, integral form). [Eva10, page-625] Let $f(\cdot)$ be a nonnegative, summable function on $[0,T]$ such that $f(t) \leq A\int_0^t f(s)ds + B$, for a.e t and some arbitarary constants A and B. Then

$$f(t) \leq B\left(1 + At\exp(At)\right), \quad \text{a.e for } 0 \leq t \leq T. \tag{1.3.9}$$

Notations: In the sequel for the unknown function $u = u(t,x)$, (t,x) in $[0,\infty) \times \mathbb{R}^n$, we use the notations: $D_x u(t,x)$ or $\nabla_x u(t,x)$ or simply $\nabla u(t,x)$ to mean the gradient of u with respect to x, written as $\nabla u(t,x) = (u_{x_1}(t,x), u_{x_2}(t,x), \cdots, u_{x_n}(t,x)) = \left(\frac{\partial}{\partial x_1}u(t,x), \frac{\partial}{\partial x_2}u(t,x), \cdots, \frac{\partial}{\partial x_n}u(t,x)\right)$ and $u_t(t,x) = \frac{\partial}{\partial t}u(t,x)$, the partial derivative of u with respect to t. We also use $|x|$ to denote the usual Euclidean norm and $x \cdot y$ is the Euclidean scalar product.

2. Viscosity Solutions

2.1 Why viscosity solutions

It is known that first order partial differential equations such as the HJE, because of their non-linearity, may not have smooth solution. The solutions obtained using the method of characteristics may not be unique or may not be regular enough. Thus, it is important to consider solutions to the HJE for which the existence, consistency and uniqueness hold true. The notion of viscosity solutions to non-linear first order PDE such as the HJE was introduced by Michael G. Crandall and Pierre-Louis Lions [CEL84] though some previous works by Evans were closely related to the concept of the viscosity solutions[CEL84]. The original uniqueness proof of viscosity solutions for non-linear first order PDE was written by Michael G. Crandall and Pierre L. Lions [CL83]. Also, Michael G. Crandall, Lawrence C. Evans and Pierre L. Lions recast the uniqueness proof in [CEL84] and it was revisited in [GCIL87] by M. G. Crandall, P. L. Lions and Hitoshi Ishii.

In this chapter, we present the theory of special types of weak solutions to HJE, called viscosity solutions, in which the consistency in the sense of classical solutions, uniqueness and some other properties hold. We here after consider an initial value problem for a non linear first order Hamilton-Jacobi PDE of the type

$$u_t(t,x) + H(x, \nabla u(t,x)) = 0, \quad \text{in} \quad (0,\infty) \times \mathbb{R}^n, \tag{2.1.1}$$

with initial condition

$$u(0,x) = \rho(x), \quad (x \in \mathbb{R}^n). \tag{2.1.2}$$

Where, $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the Hamiltonian and $\rho : \mathbb{R}^n \to \mathbb{R}$ the initial function are given.

$u : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is the unknown function, and $\nabla u = \nabla_x u = \left(\dfrac{\partial u}{\partial x_1}, ..., \dfrac{\partial u}{\partial x_n}\right)$ is the gradient of u.

From now on, we assume that the Hamiltonian H and the initial function ρ are continuous functions, and we write IVP-HJE for initial value problem of Hamilton-Jacobi equation.

2.1.1 Definition (M.G.Crandall, L.C.Evans, P.L.Lions)**.** [Eva10, page 542] A bounded, uniformly continuous function u is said to be a viscosity solution of the IVP-HJE (2.1.1) -(2.1.2) provided that the following conditions are satisfied

i) $u(0,x) = \rho(x), \quad x \in \mathbb{R}^n$,

ii) for any $v \in C^\infty((0,\infty) \times \mathbb{R}^n)$,

a) If $u - v$ attains a local minimum at a point (t_0, x_0) in $(0,\infty) \times \mathbb{R}^n$, then

$$v_t(t_0,x_0) + H(x_0, \nabla v(t_0,x_0)) \geq 0. \tag{2.1.3}$$

b) If $u - v$ attains a local maximum at a point (t_0, x_0) in $(0,\infty) \times \mathbb{R}^n$, then

$$v_t(t_0,x_0) + H(x_0, \nabla v(t_0,x_0)) \leq 0. \tag{2.1.4}$$

2.1.2 Remark. i) In the case the assertion $(ii-a)$ happens, we say that v is viscosity supersolution and in the case $(ii-b)$ holds, we say that v is viscosity subsolution [Gom09].

ii) A viscosity solution is a continuous function.

iii) If the solution u is Lipschitz continuous in $(0,\infty)\times\mathbb{R}^n$, then u is called a Lipschitz viscosity solution.

iv) Some other references define viscosity solution using sub differential and super differential, (see e.g., [Bre01] and [Gom09]).

Motivation for the definition of Viscosity solutions: [Eva10] The existence of viscosity solution has been obtained using vanishing viscosity method. The approach is to consider parabolic approximation problem, (called viscous equation), of the form

$$u_t^\epsilon(t,x)+H(x,\nabla u^\epsilon(t,x))=\epsilon\Delta u^\epsilon(t,x), \quad \text{in} \quad (0,\infty)\times\mathbb{R}^n, \tag{2.1.5}$$

$$u^\epsilon(0,x)=\rho(x), \qquad x\in\mathbb{R}^n, \tag{2.1.6}$$

for a given $\epsilon>0$. Then the claim is that as the viscosity coefficient ϵ tends to 0, the solution u^ϵ of (2.1.5)-(2.1.6) will converge to some weak solution u of (2.1.1)-(2.1.2). This is called the method of vanishing viscosity. The justification behind this is explained in [Eva10, page-540] that if we take a sequence $\{u^\epsilon\}_{\epsilon>0}$ which is bounded and equicontinuous on some compact subset of $[0,\infty)\times\mathbb{R}^n$, then by Arzela-Ascoli compactness criterion, there is a subsequence $\{u^{\epsilon_j}\}_{j=1}^\infty$ and a limit function $u\in C\left([0,\infty)\times\mathbb{R}^n\right)$ such that

$$u^{\epsilon_j}\longrightarrow u \quad \text{locally uniformly} \quad \text{in} \quad [0,\infty)\times\mathbb{R}^n. \tag{2.1.7}$$

2.1.3 Proposition. [Eva10], [Gom09] Let u^ϵ be a family of solutions of (2.1.5)-(2.1.6) such that the sequence $u^\epsilon\to u$ uniformly, as $\epsilon\to 0$. Then u is a viscosity solution of (2.1.1)-(2.1.2).

Proof. We prove for the case of local minimum. The case of local maximum can be found in [Eva10, page 541-543] and [Gom09, page 92].

Let $v\in C^\infty\left((0,\infty)\times\mathbb{R}^n\right)$ and assume that $u-v$ has strict local minimum at some point (t_0,x_0) in $(0,\infty)\times\mathbb{R}^n$. That is,

$$(u-v)(t_0,x_0)<(u-v)(t,x), \text{ for all } (t,x) \text{ sufficiently close to } (t_0,x_0), \text{ but } (t,x)\neq(t_0,x_0). \tag{2.1.8}$$

Using (2.1.7), we claim that $\forall\epsilon_j>0, \quad \exists\,(t_{\epsilon_j},x_{\epsilon_j})\in(0,\infty)\times\mathbb{R}^n$ such that

$$u^{\epsilon_j}-v \text{ has a local minimum at } (t_{\epsilon_j},x_{\epsilon_j}) \tag{2.1.9}$$

and

$$(t_{\epsilon_j},x_{\epsilon_j})\longrightarrow(t_0,x_0), \quad \text{as } j\longrightarrow\infty, \tag{2.1.10}$$

(bear in mind that $\epsilon_j\longrightarrow 0$, as $j\longrightarrow\infty$). Since $u^{\epsilon_j}-v$ has a local minimum and is differentiable at $(t_{\epsilon_j},x_{\epsilon_j})$, it holds that

$$\begin{cases} u_t^{\epsilon_j}(t_{\epsilon_j},x_{\epsilon_j})=v_t(t_{\epsilon_j},x_{\epsilon_j}) \\ \nabla u^{\epsilon_j}(t_{\epsilon_j},x_{\epsilon_j})=\nabla v(t_{\epsilon_j},x_{\epsilon_j}). \end{cases} \tag{2.1.11}$$

And we have

$$-\epsilon_j\Delta u^{\epsilon_j}(t_{\epsilon_j},x_{\epsilon_j})=-u^{\epsilon_j}(t_{\epsilon_j},x_\epsilon)-H(x_{\epsilon_j},\nabla u^{\epsilon_j}(t_{\epsilon_j},x_\epsilon))\leq-\epsilon_j\Delta v(t_{\epsilon_j},x_{\epsilon_j}). \tag{2.1.12}$$

Thus,

$$\begin{aligned} v_t(t_{\epsilon_j}, x_{\epsilon_j}) + H(x_{\epsilon_j}, \nabla v^{\epsilon_j}(t_{\epsilon_j}, x_{\epsilon_j})) &= u_t(t_{\epsilon_j}, x_{\epsilon_j}) + H(x_{\epsilon_j}, \nabla u^{\epsilon_j}(t_{\epsilon_j}, x_{\epsilon_j})) \\ &= \epsilon_j \Delta u^{\epsilon_j}(t_{\epsilon_j}, x_{\epsilon_j}), \qquad \text{from (2.1.11)} \\ &\geq \epsilon_j \Delta v(t_{\epsilon_j}, x_{\epsilon_j}), \qquad \text{from (2.1.12)}. \end{aligned}$$

Now by continuity of the functions and using (2.1.10), letting $j \longrightarrow \infty$ yields

$$v_t(t_0, x_0) + H(x_0, \nabla v(t_0, x_0)) \geq 0. \tag{2.1.13}$$

That is, if $u - v$ has a strict local minimum at (t_0, x_0), then $v_t(t_0, x_0) + H(x_0, \nabla v(t_0, x_0)) \geq 0$. In the case, $u - v$ has a local minimum, (not necessary strict local minimum), at (t_0, x_0), we set

$$\tilde{v}(t, x) = v(t, x) - \delta\left(|x - x_0| + |t - t_0|\right), \quad \text{for} \quad \delta > 0.$$

Then

$$(u - \tilde{v})(t_0, x_0) = u(t_0, x_0) - \tilde{v}(t_0, x_0) = u(t_0, x_0) - v(t_0, x_0) = (u - v)(t_0, x_0).$$

But, $(u - \tilde{v})(t_0, x_0) < (u - \tilde{v})(t, x)$ for all (t, x) sufficiently close to (t_0, x_0) with $(t, x) \neq (t_0, x_0)$. So, $u - \tilde{v}$ has strict local minimum at (t_0, x_0) and thus, the inequality (2.1.13) holds for $\tilde{v}$.

Now, suppose $u - v$ attains a strict local maximum at $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^n$, proceeding as above, but reversing the inequalities and setting

$$\tilde{v}(t, x) = v(t, x) + \delta\left(|x - x_0| + |t - t_0|\right), \quad \text{for} \quad \delta > 0, \tag{2.1.14}$$

we get (see, [Eva10, page 541-542])

$$v_t(t_0, x_0) + H(x_0, \nabla v(t_0, x_0)) \leq 0. \tag{2.1.15}$$

Therefore, the results in (2.1.13) and (2.1.15) enables us to conclude that u is a viscosity solution of the initial value problem of Hamilton-Jacobi Equation (2.1.1) -(2.1.2) when u^ϵ is a solution of the Viscous equation (2.1.6). □

2.2 Consistency of viscosity solutions

The next question, one can raise is consistency of a viscosity solution with a classical solution. Under what circumstance will be a classical solution a viscosity solution and when does a viscosity solution solves the IVP-HJE?

This section answers these questions. We begin with a lemma that proves a C^1 classical solution of the IVP-HJE is a viscosity solution.

2.2.1 Lemma. [Gom09] Let u be bounded, uniformly continuous and $u \in C^1((0, \infty) \times \mathbb{R}^n)$. If u solves (2.1.1)-(2.1.2), then it is a viscosity solution of (2.1.1)-(2.1.2).

Proof. Clearly, $u(0, x) = \rho(x), \quad x \in \mathbb{R}^n$ since u solves the IVP-HJE (2.1.1)-(2.1.2).

Let $v \in C^\infty((0, \infty) \times \mathbb{R}^n)$ be a smooth function and $u - v$ attains a local minimum at some point (t_0, x_0) in $(0, \infty) \times \mathbb{R}^n$. Since $u \in C^1((0, \infty) \times \mathbb{R}^n)$ and $v \in C^\infty((0, \infty) \times \mathbb{R}^n)$, we have

$$\begin{cases} (u - v)_t(t_0, x_0) = 0 \\ \nabla(u - v)(t_0, x_0) = 0, \end{cases} \Longrightarrow \begin{cases} u_t(t_0, x_0) = v_t(t_0, x_0) \\ \nabla u(t_0, x_0) = \nabla v(t_0, x_0). \end{cases} \tag{2.2.1}$$

By hypothesis u solves (2.1.1)-(2.1.2) for all (t,x) in $(0,\infty)\times\mathbb{R}^n$, particularly at (t_0,x_0) in $(0,\infty)\times\mathbb{R}^n$, we have

$$u_t(t_0,x_0)+H(x_0,\nabla u(t_0,x_0))=0. \tag{2.2.2}$$

Combining (2.2.1) and (2.2.2), we get

$$v_t(t_0,x_0)+H(x_0,\nabla v(t_0,x_0))=u_t(t_0,x_0)+H(x_0,\nabla u(t_0,x_0))=0.$$

So, $v_t(t_0,x_0)+H(x_0,\nabla v(t_0,x_0))\geq 0$. Thus, by definition u is super viscosity solution of the IVP-HJE (2.1.1)-(2.1.2).

Similarly, consider $v\in C^\infty((0,\infty)\times\mathbb{R}^n)$ and assume that $u-v$ attains a local maximum at some point $(t_0,x_0)\in(0,\infty)\times\mathbb{R}^n$. Proceeding as above, we arrive at

$$v_t(t_0,x_0)+H(x_0,\nabla V(t_0,x_0))\leq 0.$$

Hence, u is a sub viscosity solution of the IVP-HJE (2.1.1)-(2.1.2). Therefore, u is a viscosity solution. □

2.2.2 Theorem (consistency of viscosity solutions)**.** *[Eva10]*

If $u:[0,\infty)\times\mathbb{R}^n$ is a viscosity solution and u is differentiable at some point (t_0,x_0) in $(0,\infty)\times\mathbb{R}^n$, then $u_t(t_0,x_0)+H(x_0,\nabla u(t_0,x_0))=0$. That is, if u is a viscosity solution, then $u_t(t,x)+H(x,\nabla u(t,x))=0$ at all points (t,x) in $(0,\infty)\times\mathbb{R}^n$ where u is differentiable.

In order to prove this theorem, we first need a lemma and some notions on mollifiers.

2.2.3 Lemma. [Eva10] If $u:\mathbb{R}^n\to\mathbb{R}$ is continuous function and is also differentiable at some point $x_0\in\mathbb{R}^n$, then there is a function $\phi\in C^1(\mathbb{R}^n)$ such that $u(x_0)=\phi(x_0)$ and $u-\phi$ attains a strict local maximum at x_0.

Proof. We have to construct a function $\phi\in C^1(\mathbb{R}^n)$ that satisfies $u(x_0)=\phi(x_0)$, and

$(u-\phi)(x_0)>(u-\phi)(x),\quad \forall x\in\mathbb{R}^n$. For simplicity, let's take $x_0=0\in\mathbb{R}^n$ and $u(0)=\nabla u(0)=0$. For the case $x_0\neq 0$, we can redefined $u(x)$ appropriately. We will see it at the end of the proof.

By definition of differentiability of u at $x_0=0$, we can write

$$u(x+x_0)=u(x_0)+(x-x_0)\cdot\nabla u(x_0)+|x-x_0|\varepsilon_1(x), \tag{2.2.3}$$

where, $\varepsilon_1:\mathbb{R}^n\to\mathbb{R}$ is continuous function such that $\varepsilon_1(x_0)=\varepsilon_1(0)=0$. But, with $x_0=0$, $u(0)=\nabla u(0)=0$, we have

$$u(x)=|x|\varepsilon_1(x). \tag{2.2.4}$$

Now consider a closed ball $\bar{B}(0,r)=\bar{B}(r)\subset\mathbb{R}^n,\quad r\geq 0$ and define $\varepsilon_2:[0,\infty)\to[0,\infty)$ by:

$$\varepsilon_2(r):=\max_{x\in\bar{B}(r)}\{|\varepsilon_1(x)|\}. \tag{2.2.5}$$

Then we can observe that this function is well defined since ε_1 is continuous and $\bar{B}(r)$ is closed ball. We can easily see that ε_2 is bounded, continuous function since maximum of continuous function, if it exists, is continuous. Also, $\varepsilon_2(0)=0$, since $\varepsilon_1(0)=0$.

ε_2 is non decreasing. Because, let $r_1,\ r_2 \in [0,\infty)$ such that $r_1 \le r_2$. Then $\bar{B}(r_1) \subseteq \bar{B}(r_2)$. This implies $\max_{x\in\bar{B}(r_1)}\{|\varepsilon_1(x)|\} \le \max_{x\in\bar{B}(r_2)}\{|\varepsilon_1(x)|\}$. so, $\varepsilon_2(r_1) \le \varepsilon_2(r_2)$. Also, ε_2 is integrable on $[|x|, 2|x|]$, since it is continuous in this closed interval. Let's construct

$$\phi(x) := \int_{|x|}^{2|x|} \varepsilon_2(r)dr + |x|^2, \quad r \in [|x|, 2|x|], \quad x \in \mathbb{R}^n. \tag{2.2.6}$$

Then $\phi(0) = \int_0^0 \varepsilon_2(r)dr + |0|^2 = 0$. So, $u(0) = \phi(0)$. Since $|x| \le r \le 2|x|$ and ε_2 is non-decreasing, we have

$$\int_{|x|}^{2|x|} \varepsilon_2(|x|)dr + |x|^2 \le \int_{|x|}^{2|x|} \varepsilon_2(r)dr + |x|^2 \le \int_{|x|}^{2|x|} \varepsilon_2(2|x|)dr + |x|^2.$$

This implies $|x|\varepsilon_2(|x|) \le \phi(x) \le |x|\varepsilon_2(2|x|), \quad \forall r \in [|x|, 2|x|]$. (2.2.7)

Thus, $|x|\varepsilon_2(|x|) - \phi(x) \le 0, \quad \forall r \in [|x|, 2|x|]$. (2.2.8)

From (2.2.7), we have $0 \le \nabla\phi(0) \le 0 \ . \Longrightarrow \nabla\phi(0) = 0$.

Hence, we get $u(0) = \phi(0) = \nabla\phi(0) = 0$. So, $\phi \in C^1$ at $x_0 = 0$.

Now for $x \in \mathbb{R}^n \setminus \{0\}$, using fundamental theorem of calculus for function of several variables, we have

$$\nabla\phi(x) = \nabla\left(\int_{|x|}^{2|x|} \varepsilon_2(r)dr\right) + \nabla|x|^2 = 2\frac{x}{|x|}\varepsilon_2(2|x|) - \frac{x}{|x|}\varepsilon_2(|x|) + 2x, \quad \text{exists.}$$

So, $\phi \in C^1(\mathbb{R}^n)$. Now we left to show that $u - \phi$ has strict local maximum at $x_0 = 0$.

For $x \ne x_0 = 0$, but sufficiently close to $x_0 = 0$, from (2.2.4) and (2.2.6) we obtain

$$\begin{aligned}
u(x) - \phi(x) &= |x|\varepsilon_1(x) - \left(\int |x|^{2|x|}\varepsilon_2(r)dr + |x|^2\right) \\
&= |x|\varepsilon_1(x) - \int |x|^{2|x|}\varepsilon_2(r)dr - |x|^2 \\
&< |x|\varepsilon_2(|x|) - \int |x|^{2|x|}\varepsilon_2(r)dr - |x|^2, \quad \text{from (2.2.5)} \\
&< -|x|^2, \quad \text{from (2.2.8)} \\
&< 0.
\end{aligned}$$

That is, for x sufficiently close to $x_0 = 0$, we get $(u - \phi)(x) < (u - \phi)(0)$.

Thus, $u - \phi$ has a strict local maximum at $x_0 = 0$, as required.

When $x_0 \ne 0$, we consider

$$\tilde{u}(x) := u(x + x_0) - u(x_0) - x \cdot \nabla u(x_0), \quad \text{in place of u.} \tag{2.2.9}$$

Then observe that when $x = 0$, we get $\tilde{u}(x) = \tilde{u}(0) = 0$, and hence we will get all results we showed above for any $x_0 \in \mathbb{R}^n$. This completes the proof of the lemma. □

Now we are in a position to prove the consistency theorem.

Proof of theorem 2.2.2 :

Proof. By hypothesis, u is a viscosity solution of the IVP-HJE (2.1.1)-(2.1.2).

That is, for $v \in c^{\infty}((0,\infty)\times\mathbb{R}^n)$:

$$u - v \text{ has local minimum at}(t_0,x_0) \Longrightarrow v_t(t_0,x_0) + H(x_0,\nabla v(t_0,x_0)) \geq 0, \text{and} \tag{2.2.10}$$

$$u - v \text{ has local maximum at}(t_0,x_0) \Longrightarrow v_t(t_0,x_0) + H(x_0,\nabla v(t_0,x_0)) \leq 0. \tag{2.2.11}$$

Also it is given that u is differentiable at (t_0,x_0). We want to show that u solves (2.1.1)-(2.1.2) at (t_0,x_0).

We can denote $(0,\infty)\times\mathbb{R}^n$ as $\mathbb{R}^{n+1}$ since we can take $\mathbb{R}\times\mathbb{R}^n \cong \mathbb{R}^{n+1}$. Then u is continuous in $\mathbb{R}^{n+1}$ and differentiable at some point $(t_0,x_0)\in\mathbb{R}^{n+1}$. Thus, replacing $x_0\in\mathbb{R}^n$ by $(t_0,x_0)\in\mathbb{R}^{n+1}$ in lemma 2.2.3, there is $v\in C^1$ such that

$$u - v \text{ has a strict local maximum at } (t_0,x_0). \tag{2.2.12}$$

To use the definition of viscosity solution we have to regularize v. That is, we have to smooth (mollify) v. Mollifiers [Sho11], [Eva10, page-629] are smooth functions with special properties used to create sequence of smooth functions which approximates a non-smooth function using convolution. Let $\Omega\subset\mathbb{R}^n$ be open subset and $v:\Omega\longrightarrow\mathbb{R}$ be locally integrable, then we define the mollification of v in terms of the standard (usual) mollifier η_ϵ for, $\epsilon>0$, by

$$\begin{aligned} v^\epsilon(x) := \eta_\epsilon 9x) * v(x) &:= \int_\Omega \eta_\epsilon(x-y)v(y)dy \\ &= \int_{\bar{B}(0,\epsilon)} v(x-y)\eta_\epsilon(y)dy, \quad \forall\, x\in\Omega: \ \text{dist}(x,\partial\Omega) > \epsilon. \end{aligned} \tag{2.2.13}$$

where, $\eta\in C^\infty(\mathbb{R}^n)$ defined by

$$\eta(x) := \begin{cases} Cexp\left(\dfrac{1}{|x|^2-1}\right), & \text{when } |x|<1 \\ 0, & \text{when } |x|\geq 1, \end{cases}$$

the constant $C>0$ is selected so that $\int_{\mathbb{R}^n}\eta dx = 1$, and $\eta_\epsilon(x) := \epsilon^{-n}\eta\left(\dfrac{x}{\epsilon}\right), \quad \forall\epsilon>0.$

If $v\in C(\Omega)$, then we have (see, [Eva10, page-629] and [Sho11]) that

$$\begin{cases} a) \ \ v^\epsilon \in C^\infty \\ b) \ \ v^\epsilon \to v \quad \text{uniformly on compact sub set of } \Omega \\ c) \ \ v^\epsilon_t \to v_t \quad \text{uniformly on compact sub set of } \Omega \\ d) \ \ \nabla v^\epsilon \to \nabla v \quad \text{uniformly near } (t_0,x_0). \end{cases} \tag{2.2.14}$$

Since (2.2.12) holds and by continuity of $u-v^\epsilon$, for sufficiently small small $\epsilon>0$,

$$u - v^\epsilon \ \text{ has a maximum at some point } \ (t_\epsilon,x_\epsilon) \ \text{ such that } \ (t_\epsilon,x_\epsilon)\to(t_0,x_0) \ \text{ as } \ \epsilon\to 0. \tag{2.2.15}$$

Since u is a viscosity solution at (t_0,x_0) and using (2.2.15), we get

$$v^\epsilon_t(t_\epsilon,x_\epsilon) + H(x_\epsilon,\nabla v^\epsilon(t_\epsilon,x_\epsilon)) \leq 0. \tag{2.2.16}$$

Letting $\epsilon \to 0$ and using equations (2.2.14) and (2.2.15), equation (2.2.16) becomes

$$v_t(t_0, x_0) + H(x_0, \nabla v(t_0, x_0)) \leq 0. \tag{2.2.17}$$

Now since $u - v$ is differentiable and has local maximum at (t_0, x_0), we obtain

$$\begin{cases} u_t(t_0, x_0) = v_t(t_0, x_0) \\ \nabla u(t_0, x_0) = \nabla v(t_0, x_0). \end{cases} \tag{2.2.18}$$

Plugging this into (2.2.17), we get

$$u_t(t_0, x_0) + H(x_0, \nabla u(t_0, x_0)) \leq 0. \tag{2.2.19}$$

Similarly applying lemma (2.2.3) to $-u$ in $\mathbb{R}^{n+1}$, there is $v \in C^1$ such that

$$u - v \text{ has a srtict local minimum at some point } (t_0, x_0). \tag{2.2.20}$$

Proceeding as above, but reversing inequalities, we arrive at

$$u_t(t_0, x_0) + H(x_0, \nabla u(t_0, x_0)) \geq 0. \tag{2.2.21}$$

Combining (2.2.19) and (2.2.21), we obtain $u_t(t_0, x_0) + H(x_0, \nabla u(t_0, x_0)) = 0$. This implies that u solves IVP-HJE (2.1.1)-(2.1.2) at the point (t_0, x_0). This completes the proof of theorem (2.2.2). □

2.2.4 Corollary. If u is a viscosity solution of (2.1.1) and is Lipschitz continuous, then $u_t(t, x) + H(x, \nabla u(t, x)) = 0$ almost every where.

Proof. u is Lipschitz continuous $\implies$ u is differentiable a.e, by Hans Rademacher theorem [Eva10]. Then by theorem 2.2.2, $u_t(t, x) + H(x, \nabla u(t, x)) = 0$. □

2.3 Uniqueness of viscosity solutions

In the previous section, we investigated the consistency of viscosity solutions to the defined IVP-HJE. Now, we develop the uniqueness of viscosity solutions. We redefine the IVP-HJE for a fixed terminal time $T > 0$ as:

$$u_t(t, x) + H(x, \nabla u(t, x)) = 0, \quad \text{in} \quad (0, T] \times \mathbb{R}^n, \tag{2.3.1}$$

$$u(0, x) = \rho(x), \quad x \in \mathbb{R}^n, \tag{2.3.2}$$

where, $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\rho : \mathbb{R}^n \to \mathbb{R}$ and $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ are as described in (2.1.1)-(2.1.2).

Accordingly, we remodify the definition of viscosity solution on $[0, T] \times \mathbb{R}^n$ as follows:

2.3.1 Definition. A bounded, uniformly continuous function $u : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is called a viscosity solution of The IVP-HJE (2.3.1)-(2.3.2) when the following conditions are satisfied

i) $u(0, x) = \rho(x), \quad x \in \mathbb{R}^n$.

ii) for each $v \in C^\infty((0, T) \times \mathbb{R}^n)$, we have

a) If $u - v$ attains a local minimum at some point $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$, then

$$v_t(t_0, x_0) + H(x_0, \nabla v(t_0, x_0)) \geq 0. \tag{2.3.3}$$

b) If $u - v$ attains a local maximum at some point $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$, then

$$v_t(t_0, x_0) + H(x_0, \nabla v(t_0, x_0)) \leq 0. \tag{2.3.4}$$

Before we state and prove the uniqueness of viscosity solutions, we prove a lemma as a preliminary. This lemma investigates the behaviour of a viscosity solution at terminal time $t = T$. Of course the problem of the initial time $t = 0$ is solved by providing the initial function $u(0, x) = \rho(x)$.

2.3.2 Lemma (Etrema at the terminal time T)**.** [Eva10] Let u is a viscosity solution of (2.3.1)-(2.3.2) in $[0, T] \times \mathbb{R}^n$ and $v \in C^\infty$. If $u - v$ has a local minimum (maximum) at some point (t_0, x_0) in $(0, T] \times \mathbb{R}^n$, then

$$v_t(t_0, x_0) + H(x_0, \nabla v(t_0, x_0)) \geq 0 (\leq 0), \quad \text{respectively.} \tag{2.3.5}$$

Notice that the fact we get from this lemma different from the definition (2.3.1) is the inequalities hold true when $t_0 = T$.

Proof. It is obvious from definition (2.3.1) that if $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$, then the inequalities (2.3.5) hold. We remain to show if the inequalities hold when $t_0 = T$.

We show separately when $u - v$ has local minimum and local maximum. Suppose $(T, x_0) \in (0, T] \times \mathbb{R}^n$ is a local minimum point of $u - v$. For $\epsilon > 0$ sufficiently small, define

$$\tilde{v}(t, x) := v(t, x) - \frac{\epsilon}{T - t}, \quad \forall\ (t, x) \in (0, T) \times \mathbb{R}^n, \tag{2.3.6}$$

(this is possible by continuity of $u - v$ in $(0, T) \times \mathbb{R}^n$). Then for $0 < t < T$, $\tilde{v}(t, x) \to v(x)$, as $\epsilon \to 0$ and we have

$$\begin{cases} \tilde{v}_t(t, x) = v_t(t, x) - \dfrac{\epsilon}{(T - t)^2} \\ \nabla \tilde{v}(t, x) = \nabla v(t, x). \end{cases} \tag{2.3.7}$$

This implies

$$\tilde{v} \in C^\infty((0, T) \times \mathbb{R}^n). \tag{2.3.8}$$

Now since v is continuous, then $\tilde{v}$ is continuous for $t \neq T$, and since $u - v$ has a local minimum at $(T, x_0) \in (0, T] \times \mathbb{R}^n$, then

$$u - \tilde{v} \ \text{ has a local minimum at some interior point } \ (t_\epsilon, x_\epsilon) \ \text{ for small enough } \ \epsilon > 0. \tag{2.3.9}$$

$$\text{such that } \ 0 < t_\epsilon < T \ \text{ and } \ (t_\epsilon, x_\epsilon) \to (T, x_0) \ \text{ as } \ \epsilon \to 0. \tag{2.3.10}$$

Since u is a viscosity solution, equations (2.3.8) and (2.3.10) imply

$$\tilde{v}_t(t_\epsilon, x_\epsilon) + H(x_\epsilon, \nabla \tilde{v}(t_\epsilon, x_\epsilon)) \geq 0. \tag{2.3.11}$$

Substituting (2.3.7) in to (2.3.11), we have

$$v_t(t_\epsilon, x_\epsilon) - \frac{\epsilon}{(T - t_\epsilon)^2} + H(x_\epsilon, \nabla v(t_\epsilon, x_\epsilon)) \geq 0. \tag{2.3.12}$$

Also $\dfrac{\epsilon}{(T-t_\epsilon)^2} \to 0$ as $\epsilon \to 0$, since $T - t_\epsilon$ grows faster than ϵ or using Taylor series expansion.

Hence, (2.3.12) becomes

$$v_t(T, x_0) + H(x_0, \nabla v(T, x_0)) \geq 0, \quad \text{as required.} \tag{2.3.13}$$

On the other hand, assuming $u - v$ attains a local maximum at $(T, x_0) \in (0, T] \times \mathbb{R}^n$, we define

$$\tilde{v}(t, x) := v(t, x) + \frac{\epsilon}{T - t}, \quad \text{in} \quad (0, T) \times \mathbb{R}^n.$$

Proceeding as above, but reversing the inequality, (see, [Eva10, page 547]), we get

$$v_t(T, x_0) + H(x_0, \nabla v(T, x_0)) \leq 0, \quad \text{as required.} \tag{2.3.14}$$

Hence, the proof of the lemma. □

2.3.3 Theorem (Uniqueness of viscosity solutions). *[Eva10], [CL83], [Bre01]*

If the Hamiltonian H satisfies Lipschitz continuity conditions of the form

$$\begin{cases} |H(x,p) - H(x,q)| \leq C|p - q| \\ |H(x,p) - H(y,p)| \leq C|x - y|(1 + |p|), \end{cases} \tag{2.3.15}$$

for $x,\ y,\ ,p\ ,q \in \mathbb{R}^n$ and arbitrary constant C, then the IVP for HJE (2.3.1)-(2.3.2) has at most one viscosity solution in $[0, T] \times \mathbb{R}^n$.

Proof. The proof comes from [Eva10] and some modifications which I have learnt from [Bre01]. Suppose that the IVP for HJE has a viscosity solution. We should show that this viscosity solution is unique. We use proof by contradiction. The proof is considered in many steps.

1) Assume by contradiction that, under the assumption (2.3.15), the IVP for HJE (2.3.1)-(2.3.2) has more than one viscosity solutions in $[0, T] \times \mathbb{R}^n$.

   Let φ and $\tilde{\varphi}$ be two different viscosity solutions with the same initial condition. With out loss of generality, assume $\varphi(t', x') > \tilde{\varphi}(t', x')$ for some $(t', x') \in (0, T] \times \mathbb{R}^n$.

   Then we can find $\sigma > 0$ such that

$$\sup_{[0,T]\times\mathbb{R}^n} (\varphi - \tilde{\varphi}) =: \sigma > 0. \tag{2.3.16}$$

2) Assume that the suprimum is obtained a some point (t_0, x_0) in $(0, T] \times \mathbb{R}^n$.

   If φ and $\tilde{\varphi}$ are differentiable at such point (t_0, x_0), then we have

$$(\varphi - \tilde{\varphi})(t_0, x_0) \geq 0, \qquad \nabla\varphi(t_0, x_0) = \nabla\tilde{\varphi}(t_0, x_0)$$

   But, since φ and $\tilde{\varphi}$ are viscosity solutions we notice that

$$\begin{cases} \varphi_t(t_0, x_0) + H(x_0, \nabla\varphi(t_0, x_0)) \leq 0 \\ \tilde{\varphi}_t(t_0, x_0) + H(x_0, \nabla\tilde{\varphi}(t_0, x_0)) \geq 0 \end{cases} \Longrightarrow \begin{cases} \varphi_t(t_0, x_0) + H(x_0, \nabla\varphi(t_0, x_0)) \leq 0 \\ \tilde{\varphi}_t(t_0, x_0) + H(x_0, \nabla\varphi(t_0, x_0)) \geq 0 \end{cases}$$

   Subtracting the second inequality from the first inequality yields $\varphi_t(t_0, x_0) - \tilde{\varphi}_t(t_0, x_0) \leq 0$, which is contradiction.

   Therefore, in this case $\varphi \equiv \tilde{\varphi}$. That is, there is at most one viscosity solution.

3) We extend the above argument to a general case. When proving this, we may encounter with two technical difficulties. First, the functions φ and $\tilde{\varphi}$ may not attain their absolute maximum over the unbounded set $[0,T]\times\mathbb{R}^n$. Secondly, if there exists a point of maximum, the functions φ and $\tilde{\varphi}$ may not be differentiable at this point.

   To over come these problems, we insert a penalization term, and we use the principle of doubling the number of variables. That is, for $(t,x)\in[0,T]\times\mathbb{R}^n$, double the number of variables in to $(t,s,x,y)\in[0,T]\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^n=[0,T]^2\times\mathbb{R}^{2n}$. Now take two numbers $\epsilon>0,\ \ 0<\lambda<1$ and for each $x,y\in\mathbb{R}^n,\ \ t,s\in[0,T]$, define a function $\Theta:[0,T]^2\times\mathbb{R}^{2n}\longrightarrow\mathbb{R}$ by

$$\Theta(t,s,x,y):=\varphi(t,x)-\tilde{\varphi}(s,y)-\lambda(t+s)-\frac{1}{\epsilon^2}\left(|x-y|^2+(t-s)^2\right)-\epsilon\left(|x|^2+|y|^2\right). \quad (2.3.17)$$

   Since φ and $\tilde{\varphi}$ are bounded, there exists a constant M such that $(\varphi-\tilde{\varphi})(t,x)\le M,$ in $[0,T]\times\mathbb{R}^n$. Thus, as one of the variables goes to $+\infty,\ \ \Theta\longrightarrow-\infty$ and this implies Θ attains an absolute maximum at some point $(t_0,s_0,x_0,y_0)\in[0,T]^2\times\mathbb{R}^{2n}$. Put

$$\Theta(t_0,s_0,x_0,y_0):=\max_{[0,T]^2\times\mathbb{R}^{2n}}\Theta(t,s,x,y). \quad (2.3.18)$$

4) Let's try to see the behaviour of the point (t_0,s_0,x_0,y_0), where maximum is obtained, when ϵ tends to0. The first result, we may get is with out doubling the variables. That is, when $y=x,\ \ s=t$, we have $\Theta(t,t,x,y)=\varphi(t,x)-\tilde{\varphi}(t,x)-\lambda(2t)-2\epsilon|x|^2$. Choosing $\epsilon>0$ sufficiently small, one has

$$\Theta(t_0,s_0,x_0,y_0)=\max_{[0,T]^2\times\mathbb{R}^{2n}}\Theta(t,s,x,y)\ge\max_{[0,T]\times\mathbb{R}^n}\Theta(t,t,x,x)\ge\frac{\sigma}{2}. \quad (2.3.19)$$

   Also, we have $\Theta(t_0,s_0,x_0,y_0)\ge\Theta(0,0,0,0)=0$. This implies

$$\varphi(t_0,x_0)-\tilde{\varphi}(s_0,y_0)-\lambda(t_0+s_0)-\frac{1}{\epsilon^2}\left(|x_0-y_0|^2+(t_0-s_0)^2\right)-\epsilon\left(|x_0|^2+|y_0|^2\right)\ge 0.$$

   That is,

$$\lambda(t_0+s_0)+\frac{1}{\epsilon^2}\left(|x_0-y_0|^2+(t_0-s_0)^2\right)+\epsilon\left(|x_0|^2+|y_0|^2\right)\le\varphi(t_0,x_0)-\tilde{\varphi}(s_0,y_0). \quad (2.3.20)$$

5) At the point (t_0,t_0,x_0,x_0), we have

$$\Theta(t_0,t_0,x_0,x_0)=\varphi(t_0,x_0)-\tilde{\varphi}(t_0,x_0)-2\lambda t_0-2|x_0|^2\epsilon.$$

   From the fact that

$$\Theta(t_0,t_0,x_0,x_0)\le\Theta(t_0,s_0,x_0,y_0):=\max_{[0,T]^2\times\mathbb{R}^{2n}}\Theta(t,s,x,y),$$

   we obtain

$$\frac{1}{\epsilon^2}\left(|x_0-y_0|^2+(t_0-s_0)^2\right)\le\tilde{\varphi}(t_0,x_0)-\tilde{\varphi}(s_0,y_0)+\lambda(t_0-s_0)+\epsilon(|x_0|^2-|y_0|^2). \quad (2.3.21)$$

6) Let's introduce a function $\xi : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ that maps $(t,x) \longmapsto \Theta(t,s_0,x,y_0)$, defined as

$$\xi(t,x) := \tilde{\varphi}(s_0,y_0) + \lambda(t+s_0) + \frac{1}{\epsilon^2}\left(|x-y_0|^2 + (t-s_0)^2\right) + \epsilon\left(|x|^2 + |y_0|^2\right). \tag{2.3.22}$$

Then, $(\varphi - \xi)(t,x) = \varphi(t,x) - \tilde{\varphi}(s_0,y_0) - \lambda(t+s_0) - \frac{1}{\epsilon^2}\left(|x-y_0|^2 + (t-s_0)^2\right) - \epsilon\left(|x|^2 + |y_0|^2\right)$.

By virtue of (2.3.18), ξ has a maximum at the point (t_0,x_0) and hence the difference

$$\varphi - \xi \quad \text{has a maximum at the point} \quad (t_0,x_0). \tag{2.3.23}$$

Thus, by lemma (2.3.2) , we have

$$\xi_t(t_0,x_0) + H(x_0, \nabla\xi(t_0,x_0)) \leq 0. \tag{2.3.24}$$

But,

$$\begin{cases} \xi_t(t,x) = \lambda + \dfrac{2}{\epsilon^2}(t-s_0) \\ \nabla_x\xi(t,x) = \dfrac{2}{\epsilon^2}\left(x - y_0 + 2\epsilon x\right). \end{cases} \Longrightarrow \begin{cases} \xi_t(t_0,x_0) = \lambda + \dfrac{2}{\epsilon^2}(t_0-s_0) \\ \nabla_x\xi(t_0,x_0) = \dfrac{2}{\epsilon^2}\left(x_0 - y_0 + 2\epsilon x_0\right). \end{cases} \tag{2.3.25}$$

This allows us to write (2.3.24) as

$$\lambda + \frac{2}{\epsilon^2}(t_0-s_0) + H(x_0,\ 2\epsilon x_0 + \frac{2}{\epsilon^2}(x_0-y_0)) \leq 0. \tag{2.3.26}$$

Analogously, the mapping $\tilde{\xi} : (s,y) \longmapsto -\Theta(,t_0,s,x_0,y)$ defined by $\tilde{\xi}(s,y) := \varphi(t_0,x_0) - \lambda(t_0+s) - \frac{1}{\epsilon^2}\left(|x_0-y|^2 + (t_0-s)^2\right) - \epsilon(|x_0|^2 + |y|^2)$, has a minimum value at the point (s_0,y_0). Since $\tilde{\varphi}$ is viscosity solution (so that bounded)and $\tilde{\xi}$ is has minimum at (s_0,y_0), the difference

$$\tilde{\varphi} - \tilde{\xi} \quad \text{has a minimum value at the point} \quad (s_0,y_0), \tag{2.3.27}$$

where, $(\tilde{\varphi} - \tilde{\xi})(s,y) = \tilde{\varphi}(s,y) - \varphi(t_0,x_0) + \lambda(t_0+s) - \frac{1}{\epsilon^2}\left(|x-y|^2 + (t_0-s)^2\right) + \epsilon(|x_0|^2 + |y_0|^2)$.

Since $\tilde{\varphi}$ is a viscosity solution, in view of (2.3.27) and lemma (2.3.2), we conclude that

$$\tilde{\xi}_s(s_0,y_0) + H(y_0, \nabla\tilde{\xi}(s_0,y_0)) \geq 0. \tag{2.3.28}$$

But,

$$\begin{cases} \bar{\xi}_s(s,y) = -\lambda + \dfrac{2}{\epsilon^2}(t_0-s) \\ \nabla_y\bar{\xi}(s,y) = \dfrac{2}{\epsilon^2}\left(x_0 - y\right) - 2\epsilon y. \end{cases} \Longrightarrow \begin{cases} \bar{\xi}_s(s_0,y_0) = -\lambda + \dfrac{2}{\epsilon^2}(t_0-s_0) \\ \nabla_y\bar{\xi}(s_0,y_0) = \dfrac{2}{\epsilon^2}\left(x_0 - y_0\right) - 2\epsilon y_0. \end{cases} \tag{2.3.29}$$

Plugging this in to (2.3.28) to obtain

$$-\lambda + \frac{2}{\epsilon^2}(t_0-s_0) + H(y_0, -2\epsilon y_0 + \frac{2}{\epsilon^2}(x_0-y_0)) \geq 0. \tag{2.3.30}$$

7) Now subtracting the inequality (2.3.30) from the inequality (2.3.26) yields $2\lambda + H(x_0, 2\epsilon x_0 + \frac{2}{\epsilon^2}(x_0 - y_0)) - H(y_0, -2\epsilon y_0 + \frac{2}{\epsilon^2}(x_0 - y_0))$. This implies

$$2\lambda \leq H(y_0, -2\epsilon y_0 + \frac{2}{\epsilon^2}(x_0 - y_0)) - H(x_0, 2\epsilon x_0 + \frac{2}{\epsilon^2}(x_0 - y_0)). \tag{2.3.31}$$

Now going back to our hypothesis, since H is Lipschitz continuous, we can write

$$|H(x,p) - H(x,q)| = |H(x,p) - H(x,q) + H(x,q) - H(y,q)| \leq C(|p-q|) + C|x-y|(1+|q|)$$

So, (2.3.31) becomes

$$\begin{aligned}
2\lambda &\leq H(y_0, -2\epsilon y_0 + \frac{2}{\epsilon^2}(x_0 - y_0)) - H(x_0, 2\epsilon x_0 + \frac{2}{\epsilon^2}(x_0 - y_0)) \\
&= H(y_0, -2\epsilon y_0 + \frac{2}{\epsilon^2}(x_0 - y_0)) - H(x_0, -2\epsilon y_0 + \frac{2}{\epsilon^2}(x_0 - y_0)) \\
&\qquad + H(x_0, -2\epsilon y_0 + \frac{2}{\epsilon^2}(x_0 - y_0)) - H(x_0, 2\epsilon x_0 + \frac{2}{\epsilon^2}(x_0 - y_0)) \\
&\leq |H(y_0, -2\epsilon y_0 + \frac{2}{\epsilon^2}(x_0 - y_0)) - H(x_0, -2\epsilon y_0 + \frac{2}{\epsilon^2}(x_0 - y_0))| \\
&\qquad + |H(x_0, -2\epsilon y_0 + \frac{2}{\epsilon^2}(x_0 - y_0)) - H(x_0, 2\epsilon x_0 + \frac{2}{\epsilon^2}(x_0 - y_0))| \\
&\leq C|x_0 - y_0| \left(1 + \left|2\epsilon y_0 + \frac{2}{\epsilon^2}(x_0 - y_0)\right|\right) + C|2\epsilon y_0 - 2\epsilon x_0| \\
&= C|x_0 - y_0| \left(1 + \left|\frac{2(x_0 - y_0)}{\epsilon^2} - 2\epsilon y_0\right|\right) + 2\epsilon C|x_0 + y_0| \\
&\leq C|x_0 - y_0| \left(1 + \frac{|x_0 - y_0|}{\epsilon^2} + \epsilon(|x_0| + |y_0|)\right) + C\epsilon|x_0 + y_0|.
\end{aligned}$$

Therefore, $2\lambda \leq C|x_0 - y_0| \left(1 + \frac{|x_0 - y_0|}{\epsilon^2} + \epsilon(|x_0| + |y_0|)\right) + C\epsilon(|x_0| + |y_0|)$. (2.3.32)

The last inequality is deduced from the fact that $\left|\frac{x_0 - y_0}{\epsilon^2} - 2\epsilon y_0\right| \leq \frac{2}{\epsilon^2}|x_0 - y_0| + |-2\epsilon y_0|$ and rearranging like terms. Let's now determine estimations for terms in (2.3.32) which allows us to let $\epsilon \longrightarrow 0$. In (2.3.20), since φ and $\tilde{\varphi}$ are bounded, there is a constant M such that

$$\lambda(t_0 + s_0) + \frac{1}{\epsilon^2}\left(|x_0 - y_0|^2 + (t_0 - s_0)^2\right) + \epsilon\left(|x_0|^2 + |y_0|^2\right) \leq M.$$

This implies there are constants $M_1,\ M_2, \ldots$ such that

$$\lambda(t_0 + s_0) \leq M_1, \quad |x_0 - y_0|^2 \leq M_2\epsilon^2, \quad (t_0 - s_0)^2 \leq M_3\epsilon^2, \quad \epsilon(|x_0|^2 + |y_0|^2) \leq M_4.$$

$$|x_0 - y_0| \leq \sqrt{M_2}\,\epsilon, \quad |t_0 - s_0| \leq \sqrt{M_3}\,\epsilon, \quad |x_0| \leq \sqrt{M_5}\frac{1}{\sqrt{\epsilon}}, \quad |y_0| \leq \sqrt{M_6}\frac{1}{\sqrt{\epsilon}}. \quad \text{So,}$$

$$|x_0 - y_0| = \mathcal{O}(\epsilon),\ |t_0 - s_0| = \mathcal{O}(\epsilon),\ |x_0| = \mathcal{O}\left(\frac{1}{\sqrt{\epsilon}}\right),\ |y_0| = \mathcal{O}\left(\frac{1}{\sqrt{\epsilon}}\right);\ \ \epsilon(|x_0| + y_0|)| = \mathcal{O}(\sqrt{\epsilon}). \tag{2.3.33}$$

Using these results in (2.3.33) and letting ϵ tends to 0, the right hand side of equation (2.3.32) approaches to 0. Thus, $\lambda \leq 0$. This is a contradiction to our assumption that $0 < \lambda < 1$. Hence, we conclude that $\varphi \equiv \tilde{\varphi}$. The IVP for HJE (2.3.1)-(2.3.2) has at most one viscosity solution.

□

3. Optimal control theory

3.1 The concept

In chapter 2, we discussed viscosity solutions that have been established using the method of vanishing viscosity. Here we see viscosity solutions in a different way which has none thing to do with vanishing viscosity. We investigate a value function of an optimal control problem to built a viscosity solution. An optimal control theory deals with the problem of minimization or maximization of dynamical systems. A control problem can be any process that influences the behaviour of a dynamical system so that a required goal will be achieved and optimal controls are those controls which optimize the desired result. For example, if the desired goal is to minimize a cost function we deal with optimal control problems.

We focus on control system which can be modelled as a system of an initial Value Problem (IVP) for an Ordinary Differential Equation (ODE) of the form:

$$\begin{cases} \dot{\mathbf{x}}(s) = \mathbf{f}(\mathbf{x}(s), \boldsymbol{\mu}(s))\,; \quad t \leq s \leq T, \\ \mathbf{x}(t) = x, \quad x \in \mathbb{R}^n. \end{cases} \tag{3.1.1}$$

Where,

• $\mathbf{x}(\cdot)$ is called the solution of the system with respect to the control $\boldsymbol{\mu}(.)$ and $\mathbf{x}(s)$ is the *state* of the system at time s or simply the *state* trajectory of the system.

• We write $\mathbf{x}(s)$ for $\mathbf{x}(s; t, x, \boldsymbol{\mu})$.

• We take the controls to lie their values in a compact subset $\Omega \subseteq \mathbb{R}^m$.

• $T > 0$ is a fixed terminal time, $x \in \mathbb{R}^n$ is given initial state at some initial time $t \geq 0$.

• $\mathbf{f} : \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}$ is a function assumed to be bounded, Lipschitz continuous with respect to x and continuous with respect to $\boldsymbol{\mu}$. We consider the controls to lie in a set of admissible functions defined by

$$\mathcal{U} := \{\boldsymbol{\mu} : [0, T] \longrightarrow \Omega : \boldsymbol{\mu}(.) \ \text{ is measurable}\} \tag{3.1.2}$$

Under the assumptions we considered for the function $\mathbf{f}$, it is well known from the theory of ODE that, at least locally in time, for each control $\boldsymbol{\mu}(\cdot) \in \mathcal{U}$ the IVP for the ODE (3.1.1) admits a unique solution on $[t, T]$. We denote the solution as $s \longmapsto \mathbf{x}(s; t, x, \boldsymbol{\mu})$, which are trajectories when the control $\boldsymbol{\mu}$ ranges over the whole set $\mathcal{U}$.

3.1.1 Definition. Given a running cost $\gamma : \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}$ and a terminal cost $\rho : \mathbb{R}^n \longrightarrow \mathbb{R}$, both continuous functions, we define a corresponding cost functional (some times called Objective functional) $J : [0, T] \times \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ by

$$J[t, x, \boldsymbol{\mu}(\cdot)] := \rho(\mathbf{x}(T)) + \int_t^T \gamma(\mathbf{x}(s), \boldsymbol{\mu}(s))ds, \tag{3.1.3}$$

for fixed $x \in \mathbb{R}^n, \quad 0 \leq t \leq T$, and $\mathbf{x}(\cdot)$ solves the ODE (3.1.1).

The optimal control problem, we consider is to determine a control $\boldsymbol{\mu}^* \in \mathcal{U}$ that *minimizes* $J[t, x, \boldsymbol{\mu}(.)]$ over all admissible controls, where $\mathbf{x}(s) = \mathbf{x}(s; t, x, \boldsymbol{\mu}), \ \mathbf{x}(T) = \mathbf{x}(T; t, x, \boldsymbol{\mu})$.

We denote $J^*[t, x, \boldsymbol{\mu}] :=$ minimum of $J[t, x, \boldsymbol{\mu}(.)]$, where the minimum is taken over all controls in $\mathcal{U}$.

Assumptions: We here after assume that the functions $\mathbf{f}$, γ and ρ are bounded and Lipschitz continuous. That is, for $x, y \in \mathbb{R}^n$ and $\omega := \boldsymbol{\mu}(s) \in \Omega$, there is a constant C such that

$$\begin{cases} a) & |\mathbf{f}(x,\omega)| \leq C, \quad \text{and} \quad |\mathbf{f}(x,\omega) - \mathbf{f}(y,\omega)| \leq C|x-y|, \\ b) & |\gamma(x,\omega)| \leq C, \quad \text{and} \quad |\gamma(x,\omega) - \gamma(y,\omega)| \leq C|x-y|, \\ c) & |\rho(x,\omega)| \leq C, \quad \text{and} \quad |\rho(x) - \rho(y)| \leq C|x-y|. \end{cases} \tag{3.1.4}$$

3.2 Optimal control theory and its connection to HJE

Following the dynamic programming [Eva10], [Bre01], [Gom09] the above optimal control problem can be investigated by determining a value function.

3.2.1 Definition (Value function)**.** The Value function for the above functional cost (3.1.3) is defined by

$$u(t,x) := \inf_{\boldsymbol{\mu} \in \mathcal{U}} J[t,x,\boldsymbol{\mu}(.)] = \inf_{\boldsymbol{\mu} \in \mathcal{U}} \left[\rho(\mathbf{x}(T)) + \int_t^T \gamma((s), \boldsymbol{\mu}(s)) ds \right]. \tag{3.2.1}$$

So, the value function is the least cost. The issue here is to show that u solves a certian PDE of the HJE type, and conversely a solution of such a PDE gives us a sufficient condition for the optimality of a control $\boldsymbol{\mu}$.

3.2.2 Theorem (Optimality condition/Dynamic Programming Principle)**.** *[Eva10], [Bre01] Let $t < \tau < T$. Then the value function (3.2.1) can be computed as*

$$u(t,x) = \inf_{\boldsymbol{\mu} \in \mathcal{U}} \left[u(\tau, \mathbf{x}(\tau)) + \int_t^\tau \gamma(\mathbf{x}(s), \boldsymbol{\mu}(s) ds \right], \tag{3.2.2}$$

where, $\mathbf{x}(\cdot)$ solves the ODE (3.1.1) for each control $\boldsymbol{\mu}(\cdot)$.

Proof. Let u^τ be the right hand side of (3.2.2). We want to show that $u = u^\tau$, where $u(t,x)$ is given by (3.2.1). We divide the proof in to two steps.

1) We begin by showing $u \geq u^\tau$. Recall that by definition of infimum,

$$u(t,x) \leq \rho(\mathbf{x}(T)) + \int_t^T \gamma(\mathbf{x}(s), \boldsymbol{\mu}(s)) ds.$$

Now fix an $\epsilon > 0$ and $\boldsymbol{\mu}_1 \in \mathcal{U}$ such that $u(t,x) + \epsilon \geq J[t,x,\boldsymbol{\mu}_1] = \rho(\mathbf{x}_1(T)) + \int_t^T \gamma(\mathbf{x}_1(s), \boldsymbol{\mu}_1(s)) ds$, where $\mathbf{x}_1$ solves the ODE

$$\begin{cases} \dot{\mathbf{x}}_1(s) = \mathbf{f}(\mathbf{x}_1(s), \boldsymbol{\mu}_1(s)); \quad t < s < T, \\ \mathbf{x}_1(t) = x. \end{cases} \tag{3.2.3}$$

From the definition of $u(t,x)$ in (3.2.1), we have

$$u(\tau, \mathbf{x}_1(\tau)) = \inf_{\boldsymbol{\mu}_1 \in \mathcal{U}} \left[\rho(\mathbf{x}_1(T)) + \int_\tau^T \gamma(\mathbf{x}_1(s), \boldsymbol{\mu}_1(s)) ds \right] \tag{3.2.4}$$

$$\leq \rho(\mathbf{x}_1(T)) + \int_\tau^T \gamma(\mathbf{x}_1(s), \boldsymbol{\mu}_1(s)) ds. \tag{3.2.5}$$

Again, from the formula of u^τ we have

$$\begin{aligned} u^\tau(t,x) &\leq u(\tau,\mathbf{x}_1(\tau)) + \int_t^\tau \gamma(\mathbf{x}_1(s),\boldsymbol{\mu}_1(s)ds \\ &\leq \rho(\mathbf{x}_1(T)) + \int_t^\tau \gamma(\mathbf{x}_1(s),\boldsymbol{\mu}_1(s))ds + \int_\tau^T \gamma(\mathbf{x}_1(s),\boldsymbol{\mu}_1(s))ds \quad \text{by (3.2.5)} \\ &= \rho(\mathbf{x}_1(s)) + \int_t^T \gamma(\mathbf{x}_1(s),\boldsymbol{\mu}_1(s))ds \leq u(t,x)+\epsilon. \end{aligned}$$

Since, ϵ is arbitrary small, as ϵ tends to 0, we get $u^\tau(t,x) \leq u(t,x)$.

2) To show the reverse inequality, take $\epsilon > 0$, select a control $\boldsymbol{\mu}_2 : [t,\tau] \to \mathbb{R}$ in $\mathcal{U}$ and $\mathbf{x}_2$ such that

$$u^\tau + \epsilon \geq u(\tau,\mathbf{x}_2(\tau)) + \int_t^\tau \gamma(\mathbf{x}_2(s),\boldsymbol{\mu}_2(s))ds, \tag{3.2.6}$$

where $\mathbf{x}_2(s)$ solves the ODE

$$\dot{\mathbf{x}}_2(s) = \mathbf{f}(\mathbf{x}_2(s),\boldsymbol{\mu}_2(s)), \quad \mathbf{x}_2(t) = x, \quad t < s < \tau. \tag{3.2.7}$$

Further more, there exists a control $\boldsymbol{\mu}_3 : [\tau,T] \to \mathbb{R}$ in $\mathcal{U}$ and $\mathbf{x}_3$ such that

$$u(\tau,\mathbf{x}_3(\tau)) + \epsilon \geq J[\tau,\mathbf{x}_3(\tau),\boldsymbol{\mu}_3(.)) := \rho(\mathbf{x}_3(T)) + \int_\tau^T \gamma(\mathbf{x}_3(s),\boldsymbol{\mu}_3(s))ds, \tag{3.2.8}$$

where $\mathbf{x}_3(s)$ solves the ODE

$$\dot{\mathbf{x}}_3(s) = \mathbf{f}(\mathbf{x}_3(s),\boldsymbol{\mu}_3(s)), \quad \mathbf{x}_3(\tau) = \mathbf{x}_2(\tau), \quad \tau < s < T. \tag{3.2.9}$$

Now we define a new control $\boldsymbol{\mu}_4[t,T] \to \mathbb{R}$ in $\mathcal{U}$ by

$$\boldsymbol{\mu}_4(s) := \begin{cases} \boldsymbol{\mu}_2(s), & \text{if } \ t \leq s \leq \tau \\ \boldsymbol{\mu}_3(s), & \text{if } \ \tau \leq s \leq T, \end{cases} \tag{3.2.10}$$

and assume that $\mathbf{x}_4(s)$ solves the ODE

$$\dot{\mathbf{x}}_4(s) = \mathbf{f}(\mathbf{x}_4(s),\boldsymbol{\mu}_4(s)), \quad \mathbf{x}_4(t) = x, \quad t < s < T. \tag{3.2.11}$$

Since the IVP for ODE (3.1.1) has a unique solution, we should have

$$\mathbf{x}_4(s) := \begin{cases} \mathbf{x}_2(s), & \text{if } \ t \leq s \leq \tau \\ \mathbf{x}_3(s), & \text{if } \ \tau \leq s \leq T. \end{cases} \tag{3.2.12}$$

Then using the definition of the value function $u(t,x)$, we conclude that

$$\begin{aligned} u(t,x) &\leq J[t,x,\boldsymbol{\mu}_4] = \rho(\mathbf{x}_4(T)) + \int_t^T \gamma(\mathbf{x}_4(s),\boldsymbol{\mu}_4(s))ds \\ &= \int_t^\tau \gamma(\mathbf{x}_2(s),\boldsymbol{\mu}_2(s))ds + \int_\tau^T \gamma(\mathbf{x}_3(s),\boldsymbol{\mu}_3(s))ds + \rho(\mathbf{x}_3(T)), \quad \text{from (3.2.12)} \\ &\leq u^\tau + \epsilon - u(\tau,\mathbf{x}_(\tau)) + u(\tau,\mathbf{x}_(\tau)) + \epsilon, \quad \text{from (3.2.6), (3.2.8) and (3.2.9)} \\ &= u^\tau + 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary small, letting $\epsilon \to 0$, we arrive at $u(t,x) \leq u^\tau(t,x)$.
Therefore, $u(t,x) = u^\tau(t,x)$.

□

3.2.3 Remark. We can restate the above theorem replacing τ by $t+l$. Given $l>0$ so small that $t+l\le T$, then the value function can be computed as

$$u(t,x)=\inf_{\boldsymbol{\mu}\in\mathcal{U}}\left[u(t+l,\ \mathbf{x}(t+l))+\int_t^{t+l}\gamma(\mathbf{x}(s),\boldsymbol{\mu}(s))ds\right]. \tag{3.2.13}$$

3.2.4 Theorem. *[Eva10] Under assumptions (3.1.4), the value function given by (3.2.1) is bounded and Lipschitz continuous. That is, for all $x,\bar{x}\in\mathbb{R}^n,\ 0\le t,\bar{t}\le T$ there exists a constant K such that*

$$|u(t,x)|\le K,\qquad \textit{and}\qquad |u(t,x)-u(\bar{t},\bar{x})|\le K\left(|x-\bar{x}|+|t-\bar{t}|\right). \tag{3.2.14}$$

Proof. 1) By assumption the (3.1.4), γ and ρ are bounded and so is u since the sum of two bounded functions is bounded and the infimum of bounded function exists.

2) Let $(\bar{t},\bar{x})\in[0,T]\times\mathbb{R}^n$ and $\epsilon>0$ be given. Choose a control $\boldsymbol{\mu}_\epsilon:[0,T]\longrightarrow\Omega$ in $\mathcal{U}$ so that

$$J[\bar{t},\bar{x},\boldsymbol{\mu}_\epsilon(\cdot)]:=\int_{\bar{t}}^T\gamma(\mathbf{x}(s),\boldsymbol{\mu}_\epsilon(s))ds+\rho(\mathbf{x}(T)\le u(\bar{t},\bar{x})+\epsilon, \tag{3.2.15}$$

where, $\mathbf{x}(.)$ solves the ODE

$$\dot{\mathbf{x}}(s)=\mathbf{f}(\mathbf{x}(s),\boldsymbol{\mu}_\epsilon(s)),\quad \mathbf{x}(\bar{t})=\bar{x},\quad \bar{t}<s<T. \tag{3.2.16}$$

Using the same control $\boldsymbol{\mu}_\epsilon(\cdot)$, let $\hat{\mathbf{x}}(\cdot)$ be a unique solution in connection with a different initial data $\hat{\mathbf{x}}(t)=x$. Then $\hat{\mathbf{x}}(.)$ solves the ODE

$$\hat{\mathbf{x}}(s)=\mathbf{f}(\hat{\mathbf{x}}(s),\boldsymbol{\mu}_\epsilon(s)),\quad \hat{\mathbf{x}}(t)=x,\quad t<s<T. \tag{3.2.17}$$

3) But, $J[t,x,\boldsymbol{\mu}_\epsilon(\cdot)]=J[\bar{t},\bar{x},\boldsymbol{\mu}_\epsilon(\cdot)]+J[t,x,\boldsymbol{\mu}_\epsilon(\cdot)]-J[\bar{t},\bar{x},\boldsymbol{\mu}_\epsilon(\cdot)]$. Assuming $t<\bar{t}$ yields

$$\begin{aligned}J[t,x,\boldsymbol{\mu}_\epsilon(\cdot)]=J[\bar{t},\bar{x},\boldsymbol{\mu}_\epsilon(\cdot)]\quad&+\quad\int_t^{\bar{t}}\gamma(\hat{\mathbf{x}}(s),\boldsymbol{\mu}_\epsilon(s))ds\\&+\quad\int_{\bar{t}}^T\left[\gamma(\hat{\mathbf{x}}(s),\boldsymbol{\mu}_\epsilon(s))-\gamma(\mathbf{x}(s),\boldsymbol{\mu}_\epsilon(s))\right]ds+\rho(\hat{\mathbf{x}}(T))-\rho(\mathbf{x}(T)).\end{aligned}$$

Then, using the bounds in equation (3.1.4) (b) and (c), we have

$$J[t,x,\boldsymbol{\mu}_\epsilon(\cdot)]\le J[\bar{t},\bar{x},\boldsymbol{\mu}_\epsilon(\cdot)]+C|t-\bar{t}|+\int_{\bar{t}}^T C|\hat{\mathbf{x}}(s)-\mathbf{x}(s)|ds+C|\hat{\mathbf{x}}(T)-\mathbf{x}(T)|. \tag{3.2.18}$$

Since $\mathbf{f}$ is Lipschitz continuous, we can apply Gronwall's lemma to (3.2.16) and (3.2.17) to get

$$|\hat{\mathbf{x}}(s)-\mathbf{x}(s)|\le C\left(|t-\bar{t}|+|x-\bar{x}|\right),\quad t,\bar{t}\le s\le T. \tag{3.2.19}$$

Thus, using equation(3.2.19) in equation (3.2.18) yields

$$J[t,x,\boldsymbol{\mu}_\epsilon(\cdot)]\le J[\bar{t},\bar{x},\boldsymbol{\mu}_\epsilon(\cdot)]+C\left(|t-\bar{t}|+|x-\bar{x}|\right). \tag{3.2.20}$$

Then invoking equation (3.2.15); equation (3.2.20) reduces to

$$u(t,x)\le J[t,x,\boldsymbol{\mu}_\epsilon(.)]\le u(\bar{t},\bar{x})+\epsilon+C\left(|t-\bar{t}|+|x-\bar{x}|\right).$$

Rearranging and letting $\epsilon \to 0$, we obtain

$$u(t,x) - u(\bar{t},\bar{x}) \leq C\left(|t-\bar{t}| + |x-\bar{x}|\right). \tag{3.2.21}$$

Interchanging the roles of (t,x) and $(\bar{t},\bar{x})$ and proceeding as above, one obtains

$$u(\bar{t},\bar{x}) - u(t,x) \leq C\left(|\bar{t}-t| + |\bar{x}-x|\right). \tag{3.2.22}$$

Therefore, combining (3.2.21) and (3.2.22) enables us to conclude that

$$|u(\bar{t},\bar{x}) - u(t,x)| \leq C\left(|\bar{t}-t| + |\bar{x}-x|\right), \qquad \text{as required.}$$

□

3.3 Terminal Value Problem for Hamilton-Jacobi-Bellman Equation

In this section, we will introduce a special type of the HJE, called Hamilton-Jacobi-Bellman Equation (HJBE). And prove that the value function solves this PDE.

3.3.1 Definition (HJBE). [Eva10] A HJBE is a PDE of the form

$$u_t(t,x) + H(x, \nabla u(t,x)) = 0, \quad \text{in} \quad (0,T] \times \mathbb{R}^n, \tag{3.3.1}$$

where the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is given by

$$H(x,p) := \min_{\omega\in\Omega}\left\{p.\mathbf{f}(x,\omega) + \gamma(x,\omega)\right\}, \quad (x,p \in \mathbb{R}^n), \tag{3.3.2}$$

and Ω, γ, $\mathbf{f}$ are as defined in the previous section. Note that since $\mathbf{f}$ is continuous on compact set Ω, the minimum exists. Particularly, for $p = \nabla u$, we have

$$H(x,p) := H(x,\nabla u) = \min_{\omega\in\Omega}\left\{\nabla u.\mathbf{f}(x,\omega) + \gamma(x,\omega)\right\}. \tag{3.3.3}$$

A terminal value problem (TVP) for a HJBE is written as

$$\begin{cases} u_t(t,x) + \min_{\omega\in\Omega}\left\{\nabla u.\mathbf{f}(x,\omega) + \gamma(x,\omega)\right\} = 0, & \forall (t,x) \in (0,T) \times \mathbb{R}^n \\ u(T,x) = \rho(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{3.3.4}$$

3.3.2 Corollary. Under the assumption (3.2.1)$-(b)$, H is Lipschitz continuous in both x and p. That is, there is a constant $C \geq 0$ such that

$$\begin{cases} |H(x,p) - H(x,q)| \leq C|p-q| \\ |H(x,p) - H(y,p)| \leq C|x-y|(1+|p|). \end{cases} \tag{3.3.5}$$

Proof.
$$\begin{aligned} |H(x,p) - H(x,q)| &= \left|\min_{\omega\in\Omega}\left\{p.\mathbf{f}(x,\omega) + \gamma(x,\omega)\right\} - \min_{\omega\in\Omega}\left\{q.\mathbf{f}(x,\omega) + \gamma(x,\omega)\right\}\right| \\ &\leq |Cp - Cq| = C|p-q|. \end{aligned}$$

Similarly, we can show the second result.

3.3.3 Definition (viscosity solution of terminal value problem)**.** A bounded, uniformly continuous function u is said to be a viscosity solution of the TVP for HJBE (3.3.4) provided that

i) $u(T,x) = \rho(x), \quad x \in \mathbb{R}^n$.

ii) for any $v \in C^\infty((0,T) \times \mathbb{R}^n)$,

a) If $u - v$ attains a local minimum at a point $(t_0, x_0) \in (0,T) \times \mathbb{R}^n$, then

$$v_t(t_0,x_0) + \min_{\omega \in \Omega} \{\nabla v.\mathbf{f}(x_0,\omega) + \gamma(x_0,\omega)\} \leq 0. \tag{3.3.6}$$

b) If $u - v$ attains a local maximum at a point $(t_0, x_0) \in (0,T) \times \mathbb{R}^n$, then

$$v_t(t_0,x_0) + \min_{\omega \in \Omega} \{\nabla v.\mathbf{f}(x_0,\omega) + \gamma(x_0,\omega)\} \geq 0. \tag{3.3.7}$$

Observe that in the definition of viscosity solutions of the TVP for HJE and the IVP for HJE, the inequalities are reversed.

3.3.4 Corollary. If $u : (0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a viscosity solution of the TVP for HJBE (3.3.4), then $\psi : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $\psi(t,x) = u(T-t,\ x)$ is a viscosity solution of the IVP

$$\begin{cases} \psi_t(t,x) - H(x, \nabla\psi(t,x)) = 0, & \text{in} \quad (0,T) \times \mathbb{R}^n \\ \psi(0,x) = \rho(x), \quad x \in \mathbb{R}. \end{cases} \tag{3.3.8}$$

Proof. Clearly since u is a viscosity solution, ψ is bounded and uniformly continuous on $[0,T] \times \mathbb{R}^n$. Also, we have

i) $u(T,x) = \rho(x)$. So, at $t = 0$, $\psi(0,x) = u(T-0,x) = u(T,x) = \rho(x)$.

ii) for $v \in C^\infty$, if $u - v$ has local minimum(maximum) at (t_0, x_0), then
$v_t(t_0,x_0) + H(x_0, \nabla v(t_0,x_0)) \leq 0 (\geq 0)$, respectively, where H is given by (3.3.3), and
$u(t_0,x_0) - v(t_0,x_0) \leq (\geq) u(t,x) - v(t,x)$ for (t,x) sufficiently close to (t_0,x_0).
So, $\psi - v$ will have local minimum (maximum) at the point $(T-t_0, x_0)$. Since $u - v$ has local minimum(maximum) at (t_0,x_0) and $u - v$ is differentiable at (t_0,x_0), one obtains

$$\begin{cases} u_t(t_0,x_0) = v_t(t_0,x_0) \\ \nabla u(t_0,x_0) = \nabla v(t_0,x_0). \end{cases} \tag{3.3.9}$$

Similarly $\psi - v$ has local minimum(maximum) at $(T-t_0,x_0)$ and $\psi - v$ is differentiable at $(T-t_0,x_0)$. So

$$\begin{cases} \psi_t(T-t_0,x_0) = v_t(T-t_0,x_0) \\ \nabla\psi(T-t_0,x_0) = \nabla v(T-t_0,x_0). \end{cases} \tag{3.3.10}$$

But, it is given that $\psi(t,x) = u(T-t,x)$. Thus, we get

$$\psi_t(t,x) = -u_t(T-t,x) \ \text{ and } \ \nabla\psi(t,x) = \nabla u(T-t,x). \tag{3.3.11}$$

This equation and equation (3.3.9) implies

$$\psi_t(T-t_0,x_0) = -v_t(t_0,x_0) \ \text{ and } \ \nabla\psi(T-t_0,x_0) = \nabla v(t_0,x_0). \tag{3.3.12}$$

Using (3.3.6) and (3.3.9), $u_t(t_0, x_0) + H(x_0, \nabla u(t_0, x_0)) \leq 0$. Substituting (3.3.12) and then (3.3.10) gives

$$v_t(T - t_0, x_0) - H(x_0, \nabla v(T - t_0, x_0)) \geq 0. \tag{3.3.13}$$

Similarly, From (3.3.7) and (3.3.9), we have $u_t(t_0, x_0) + H(x_0, \nabla u(t_0, x_0)) \leq 0$. Substituting (3.3.12) and then (3.3.10) yields

$$v_t(T - t_0, x_0) - H(x_0, \nabla v(T - t_0, x_0)) \leq 0. \tag{3.3.14}$$

Hence, ψ is a viscosity solution.

□

3.3.5 Theorem (A PDE satisfied by Value function)**.** *[Eva10] The value function u defined by (3.2.1) is a unique viscosity solution of the TVP for the HJBE (3.3.4).*

Proof. We recall that, the value function u is bounded and Lipschitz continuous. To show that u is a viscosity solution of the TVP-HJBE, we should show that

i) $u(T, x) = \rho(x)$ and

ii) For $v \in C^\infty((0, T) \times \mathbb{R}^n)$, we have to show that

(a) if $u - v$ attains a local minimum at $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$, then

$$v_t(t_0, x_0) + \min_{\omega \in \Omega} \{\nabla v(t_0, x_0).\mathbf{f}(x_0, \omega) + \gamma(x_0, \omega)\} \leq 0. \tag{3.3.15}$$

b) if $u - v$ attains a local maximum at $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$, then

$$v_t(t_0, x_0) + \min_{\omega \in \Omega} \{\nabla v(t_0, x_0).\mathbf{f}(x_0, \omega) + \gamma(x_0, \omega)\} \geq 0. \tag{3.3.16}$$

To prove (i): Equation(3.2.1) implies

$$\begin{aligned} u(T, x) &= \inf_{\boldsymbol{\mu}(.)} \left[\rho(\mathbf{x}(T)) + \int_T^T \gamma(\mathbf{x}(s), \boldsymbol{\mu}(s)) ds\right] \\ &= \inf_{\boldsymbol{\mu}(.)} (\rho(\mathbf{x}(T) + 0) = \rho(x), \quad \text{since} \quad \rho(\mathbf{x}(T)) = \rho(x). \end{aligned}$$

To prove (a): Assume that $u - v$ attains local minimum at (t_0, x_0). This implies $u_t(t_0, x_0) = v_t(t_0, x_0)$ and $\nabla u(t_0, x_0) = \nabla v(t_0, x_0)$.

We use proof by contradiction. Suppose (3.3.15) does not hold true. That is,

$$v_t(t_0, x_0) + \min_{\omega \in \Omega} \{\nabla v(t_0, x_0).\mathbf{f}(x_0, \omega) + \gamma(x_0, \omega)\} > 0.$$

Thus, there is a control value $\omega \in \Omega$, some $\beta > 0$ and $\delta > 0$ such that

$$v_t(t, x) + \nabla v(t, x).\mathbf{f}(x, \omega) + \gamma(x, \omega) \geq \beta > 0, \quad \forall \omega \in \Omega, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \tag{3.3.17}$$

$$\text{with the property that} \quad |x - x_0| + |t - t_0| < \delta. \tag{3.3.18}$$

Since $u - v$ has local minimum at (t_0, x_0), we can assume that

$$(u - v)(t_0, x_0) \leq (u - v)(t, x), \quad \text{for all} \quad (t, x) \quad \text{satisfying} \quad (3.3.18). \tag{3.3.19}$$

Since $\mathbf{f}$ is bounded and Lipschitz continuous so that continuous, we can select $0 < l < \delta$ so small that $|\mathbf{x}(s) - x| < \delta, \quad \forall s \in [t_0, t_0 + l]$, where $\mathbf{x}(.)$ solves the ODE

$$\dot{\mathbf{x}}(s) = \mathbf{f}(\mathbf{x}(s), \boldsymbol{\mu}(s)), \quad \mathbf{x}(t_0) = x_0, \tag{3.3.20}$$

for some control $\boldsymbol{\mu}(\cdot) \in \mathcal{U}$, such that $\boldsymbol{\mu}(s) = \omega \in \Omega$ for $t_0 \leq s \leq t_0 + l$. Since $(t_0 + l, x_0 + l)$ is sufficiently close to (t_0, x_0) satisfying (3.3.18), then (3.3.19) implies

$$u(t_0 + l, \mathbf{x}(t_0 + l)) - u(t_0, x_0) \geq v(t_0 + l, \mathbf{x}(t_0 + l)) - v(t_0, x_0) \tag{3.3.21}$$

$$= \int_{t_0}^{t_0+l} \frac{d}{ds} v(s, \mathbf{x}(s)) ds = \int_{t_0}^{t_0+l} [v_s(s, \mathbf{x}(s)) + \nabla v(s, \mathbf{s}(s)) \cdot \dot{\mathbf{x}}(s)] \, ds \tag{3.3.22}$$

$$= \int_{t_0}^{t_0+l} [v_s(s, \mathbf{x}(s)) + \nabla v(s, \mathbf{s}(s)).\mathbf{f}(\mathbf{x}(s), \omega)] \, ds, \quad \text{by (3.3.20)} \tag{3.3.23}$$

Using the optimality condition, we can write

$$u(t_0, x_0) = \inf_{\boldsymbol{\mu}(.) \in \mathcal{U}} \left[\int_{t_0}^{t_0+l} \gamma(\mathbf{x}(s), \boldsymbol{\mu}(s)) ds + u(t_0 + l, \mathbf{x}(t_0 + l)) \right]. \tag{3.3.24}$$

So, we can choose a control $\boldsymbol{\mu}(.) \in \mathcal{U}$ such that

$$u(t_0, x_0) \geq \int_{t_0}^{t_0+l} \gamma(\mathbf{x}(s), \boldsymbol{\mu}(s)) ds + u(t_0 + l, \mathbf{x}(t_0 + l)) - \frac{\beta l}{2}. \tag{3.3.25}$$

$$\Longrightarrow u(t_0, x_0) - u(t_0 + l, \mathbf{x}(t_0 + l)) + \frac{\beta l}{2} \geq \int_{t_0}^{t_0+l} \gamma(\mathbf{x}(s), \boldsymbol{\mu}(s)) ds. \tag{3.3.26}$$

Adding the left side with left side and right side with right side of equations (3.3.23) and (3.3.26), one obtains

$$\begin{aligned} \frac{\beta l}{2} &\geq \int_{t_0}^{t_0+l} \underbrace{[v_s(s, \mathbf{x}(s)) + \nabla v(s, \mathbf{x}(s)).\mathbf{f}(\mathbf{x}(s), \omega) + \gamma(\mathbf{x}(s), \boldsymbol{\mu}(s))]}_{\geq \beta} ds \\ &\geq \int_{t_0}^{t_0+l} \beta ds = \beta l, \end{aligned}$$

which is a contradiction since $\beta > 0, \quad l > 0$. This contradiction comes from our wrong assumption that (3.3.15) does not hold true. Thus, the proof of (a).

Similarly, to prove (b), we proceed as above, but reversing the inequalities.

Hence, the value function is a viscosity solution of the TVP for HJBE.

Also, in theorem (2.3.3) (uniqueness of viscosity solutions), we have proved that if there exists, a viscosity solution is unique for a HJE. Since the HJBE is a special type of HJE and the Hamiltonian H for HJBE satisfies Lipschitz condition, we conclude that the value function is a unique viscosity solution of the terminal value problem for the Hamilton-Jacobi-Bellman equation. □

4. The Hopf-Lax Formula

4.1 Motivation for Hopf-Lax formula

In this chapter, we consider the HJE when the Hamiltonian H and Lagrangian L do not depend on x, that is, when $H = H(p), \quad L = L(q), \quad (p, q \in \mathbb{R}^n)$. We present a formula that can be used to obtain a viscosity solution even if to compute the formula is not easy. We consider the IVP-HJE

$$\begin{cases} u_t(t,x) + H(\nabla u(t,x)) = 0, & \text{in} \quad (0,\infty) \times \mathbb{R}^n \\ u(0,x) = \rho(x), & x \in \mathbb{R}^n. \end{cases} \tag{4.1.1}$$

The objective of this chapter is to introduce the Hopf-Lax formula and to prove that it provides a unique viscosity solution. To go through this chapter, we first consider the following assumptions in which case the Hopf-Lax formula has been established by the two mathematicians Hopf and Lax [Van04].

4.1.1 Remark (Assumptions:). In this chapter, We assume that

i) The mapping $p \longmapsto H(p)$ is convex, (we simply say H is convex in p).
   Recall that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called convex provided that
   $$f(\tau x + (1-\tau)y) \leq \tau f(x) + (1-\tau) f(y),$$
   for all $x, y \in \mathbb{R}^n$ and each $0 \leq \tau \leq 1$.

ii) H is super linear. That is, $\lim_{|p|\to\infty} \dfrac{H(p)}{|p|} = \infty$, and

iii) The initial function $\rho : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous . That is, there is a constant $C \geq 0$ such that $|\rho(x) - \rho(y)| \leq C|x-y|$. In another words,
   $$Lip(\rho) := \sup_{x,y\in\mathbb{R}^n} \left\{ \frac{|\rho(x) - \rho(y)|}{|x-y|} \right\} < \infty, \qquad \text{for } x \neq y. \tag{4.1.2}$$

We also use the assumptions (i) and (ii) for the Lagrangian L.

Note that the results we develop in this chapter hold for general a Fenchel conjugate [Hoa13], [Van04] H^* of the Hamiltonian H, which is a generalization of Legendre transform. But, for simplicity, we will consider here only for the case $H^* = L$, the Legendre transform. We begin with calculus of variations problem of an action for L. Given $L = L(q),\ x \in \mathbb{R}^n,\ t > 0$, we introduce the action functional [Eva10]

$$S[\mathbf{w}(s)] := \int_0^t L(\dot{\mathbf{w}}(s))ds, \tag{4.1.3}$$

over admissible functions $\mathbf{w} : [0,t] \to \mathbb{R}^n$ with $\mathbf{w}(t) = x$, where, L is Legendre transform of H : given by

$$H^*(q) := L(q) := \sup_{p\in\mathbb{R}^n} \{p.q - H(p)\}, \quad (q \in \mathbb{R}^n).$$

The calculus of variation problem here is we need to minimize $S[\mathbf{w}(\cdot)]$, over all such admissible functions. Since we take in to consideration the initial condition $u(0,x)$ in the definition of our HJE, we should take care of $\mathbf{w}(0)$. Thus, we rewrite the minimization problem (4.1.3) as

$$S[\mathbf{w}(s)] := \rho(\mathbf{w}(0)) + \int_0^t L(\dot{\mathbf{w}}(s))ds. \tag{4.1.4}$$

Now using variational principle, we guess that a solution of the IVP-HJE (4.1.1) could be [Eva10]

$$u(t,x) = \inf_{\mathbf{w}\in\mathcal{U}} \left\{ \rho(y) + \int_0^t L(\dot{\mathbf{w}}(s))ds \ |\mathbf{w}(0) = y, \ \mathbf{w}(t) = x \right\}, \tag{4.1.5}$$

where infimum is taken over the admissible space $\mathcal{U} := \{\mathbf{w}(\cdot) \in C^1([0,t]) : \mathbf{w}(t) = x, \ w(0) = y\}$.

Under the assumptions in remark (4.1.1), E. Frederich Ferdinand Hopf (1902-1983) [Van04] developed a formula for a viscosity solution of the HJE (4.1.1) written as a viscosity solution. Lax [Van04] had found analogous solution $u(t,x)$ by the method of vanishing viscosity.

4.1.2 Definition. [Hoa13] Under the assumptions (4.1.1), the function $u(t,x)$ defined by the formula

$$u(t,x) = \min_{y\in\mathbb{R}^n} \left\{ \rho(y) + tL\left(\frac{x-y}{t}\right) \right\}, \quad (x \in \mathbb{R}^n, \quad t > 0), \tag{4.1.6}$$

is known as the *Hopf-Lax formula* for the IVP-HJE (4.1.1), where L is the Legendre transform of H.

4.1.3 Theorem. *[Eva10], [Van04] If $x \in \mathbb{R}^n$ and $t > 0$, then the solution $u = u(t,x)$ of the minimization problem (4.1.5) is $u(t,x)$ given by the Hopf-Lax formula (4.1.6).*

Proof. It suffices to show that $u(t,x)$ in (4.1.5) is equal to $\inf_{y\in\mathbb{R}^n} \left\{ \rho(y) + tL\left(\frac{x-y}{t}\right) \right\}$ and then this infimum is minimum.

i) Let's fix $y \in \mathbb{R}^n$ and set

$$\mathbf{w}(s) := y + \frac{s}{t}(x-y), \quad (0 \le s \le t). \tag{4.1.7}$$

Then

$$\dot{\mathbf{w}}(s) = \frac{x-y}{t} \quad \text{and} \quad \int_0^t L(\dot{\mathbf{w}}(s)ds = \int_0^t L\left(\frac{x-y}{t}\right) ds = tL\left(\frac{x-y}{t}\right). \tag{4.1.8}$$

Using (4.1.8) in (4.1.5), we get

$$u(t,x) \le \rho(y) + tL\left(\frac{x-y}{t}\right). \quad \text{Thus} \quad u(t,x) \le \inf_{y\in\mathbb{R}^n} \left\{ \rho(y) + tL\left(\frac{x-y}{t}\right) \right\} \tag{4.1.9}$$

ii) To show the reverse inequality, since $\mathbf{w}(\cdot) \in C^1$ with $\mathbf{w}(t) = x$, then by Jensen's inequality, we have

$$L\left(\frac{1}{t}\int_0^t \dot{\mathbf{w}}(s)ds\right) \le \frac{1}{t}\int_0^t L(\dot{\mathbf{w}}(s))ds.$$

This implies

$$L\left(\frac{x-y}{t}\right) \quad \le \quad \frac{1}{t}\int_0^t L(\dot{\mathbf{w}}(s))ds, \quad \text{since} \quad y = \mathbf{w}(0)$$

That is,

$$tL\left(\frac{x-y}{t}\right) \le \int_0^t L(\dot{\mathbf{w}}(s))ds. \tag{4.1.10}$$

Adding $\rho(y)$ and then taking infimum of both sides in equation (4.1.10), we obtain

$$\inf_{y\in\mathbb{R}^n}\left\{\rho(y)+tL\left(\frac{x-y}{t}\right)\right\} \leq \inf_{y\in\mathbb{R}^n}\left\{\rho(y)+\int_0^t L(\dot{\mathbf{w}}(s))ds\right\}. \tag{4.1.11}$$

$$\implies \inf_{y\in\mathbb{R}^n}\left\{\rho(y)+tL\left(\frac{x-y}{t}\right)\right\} \leq u(t,x), \quad \text{by (4.1.5).} \tag{4.1.12}$$

Therefore, using (4.1.9) and (4.1.12) we conclude that

$$u(t,x)=\inf_{y\in\mathbb{R}^n}\left\{\rho(y)+tL\left(\frac{x-y}{t}\right)\right\}. \tag{4.1.13}$$

iii) We left to show that the infimum in (4.1.3) is minimum. Putting $y=x$ in the right hand side of (4.1.3) yields

$$u(t,x)\leq tL(0)+\rho(x). \tag{4.1.14}$$

From the assumption (4.1.1)-(ii), since $\lim_{|q|\to\infty}\dfrac{L(q)}{|q|}=\infty$, by definition of limit at infinity, there exists a constant C_1 such that $L(q)\geq(|p|+1)|q|, \quad |p|,|q|\in\mathbb{R}$ whenever $|q|>C_1$. Since $Lip(\rho)\in\mathbb{R}$, we can take $|p|=Lip(\rho)=\sup_{x,y}\left\{\dfrac{|\rho(x)-\rho(y)|}{|x-y|}\right\}<\infty, \quad x,y\in\mathbb{R}^n,\ x\neq y$. So, $L(q)\geq(Lip(\rho)+1)|q|, \quad$ for $\ |q|\geq C_1$.

Thus, by taking a look over the term inside infimum in (4.1.13), if $|x-y|\geq tC_1$, we have

$$\begin{aligned}
\rho(y)+tL\left(\frac{x-y}{t}\right) &\geq \rho(y)+2(Lip(\rho)+1)|x-y| \\
&\geq \rho(x)+(Lip(\rho)+2)|x-y| \\
&\geq -tL(0)+u(t,x)+(Lip(\rho)+2)|x-y|, \qquad \text{from (4.1.14)} \\
&\geq u(t,x), \quad \text{for} \quad |x-y|\geq tC_2,
\end{aligned}$$

where $\ C_2=\max\left[C_1,\dfrac{L(0)}{Lip(\rho)+1}\right]$.

The last in equality is because, if we take for instance, $C_2=\dfrac{L(0)}{Lip(\rho)+1}$, then

$|x-y|\geq tC_2=t\dfrac{L(0)}{Lip(\rho)+1}$ and thus

$$-tL(0)+(Lip(\rho)+2)|x-y|\geq -tL(0)+(Lip(\rho)+2)\left(t\frac{L(0)}{Lip(\rho+1}\right)=t\frac{L(0)}{Lip(\rho)+1}\geq 0.$$

So, $\rho(y)+tL\left(\dfrac{x-y}{t}\right)\geq \underbrace{-tL(0))+(Lip(\rho)+2)|x-y|}_{\geq 0}+u(t,x)\geq u(t,x)$.

That is, $\rho(y)+tL\left(\dfrac{x-y}{t}\right)\geq u(t,x)$, for $|x-y|\geq tC_2$.

Hence, equation (4.1.13) reduces in to

$$u(t,x)=\min\left\{\rho(y)+tL\left(\frac{x-y}{t}\right)\right\}, \quad \forall y \ \text{ Satisfying} |x-y|\geq tC_2.$$

Hence, the proof of the theorem.

□

Next we state two lemmas without proof, where their proofs can be found in [Eva10, chapter 3] and [Hoa13].

4.1.4 Lemma. For each $x \in \mathbb{R}^n$ and $0 < s < t$, the function u defined by the Hopf-Lax formula (4.1.6) can be computed as

$$u(t,x) = \min_{y \in \mathbb{R}^n} \left\{ u(s,y) + (t-s)L\left(\frac{x-y}{t-s}\right) \right\}. \tag{4.1.15}$$

That is , to determine $u(t,.)$, we find u at time s and then use $u(s,.)$ as initial condition on the remaining time interval $[s,t]$.

Proof. [Eva10, page 125], [Hoa13] □

4.1.5 Lemma. Under the assumptions (4.1.1), the function u defined by the Hopf-Lax formula (4.1.6) is Lipschitz continuous in $(0,\infty) \times \mathbb{R}^n$. That is,

$$|u(t,x) - u(t,\bar{x})| \leq Lip(\rho)|x - \bar{x}|, \quad \text{and} \quad |u(t,x) - u(\bar{t},x| \leq C|t - \bar{t}| \quad \text{for all } x, \bar{x} \in \mathbb{R}^n,\ t, \bar{t} \in (0,\infty),$$

where,

$$C = \max\left(|L(0)|, \max_{\bar{B}(0,Lip(\rho))} |H| \right). \tag{4.1.16}$$

Proof. See [Eva10, page 127]. □

4.1.6 Corollary. i) The function u defined by Hopf-Lax formula is differentiable a.e in $(0,\infty) \times \mathbb{R}^n$.

ii) $Lip(u(t,\cdot)) \leq Lip(\rho)$.

Proof. By previous lemma, u is Lipschitz continuous. By Hans Rademacher's theorem u is differentiable a.e in $(0,\infty) \times \mathbb{R}^n$. For the second proof, use the fact that $|u(t,x) - u(t,\bar{x})| \leq Lip(\rho)|x - \bar{x}|$. □

4.2 The Hopf-Lax formula as viscosity solution

We begin with proving that when u given by the Hopf-Lax formula is differentiable, then it solves the HJE, then prove that it is a unique viscosity solution.

4.2.1 Theorem (a PDE solved by the Hopf-Lax formula)**.** *[Eva10, page 128], [Hoa13, page 9]*

The function u defined by the Hopf-Lax formula solves the IVP-HJE at all points in $(0,\infty) \times \mathbb{R}^n$ where it is differentiable. That is, if u is differentiable at (t,x), then $u_t(t,x) + H(\nabla u(t,x)) = 0$.

Proof. The proof comes from [Eva10] and some modifications which I have learnt from [Hoa13].

Fix $x \in \mathbb{R}^n,\ t > 0$. It is required to show that $u(0,x) = \rho(x)$ and $u_t(t,x) + H(\nabla u(t,x)) = 0$ at all points (t,x) in $(0,\infty) \times \mathbb{R}^n$ where u is differentiable.

i) Since the Hopf-Lax formula holds for all $y \in \mathbb{R}^n$, particularly take $y = x$ in (4.1.6), we get

$$u(t,x) \leq \rho(x) + tL(0), \quad \text{in } (0,\infty) \times \mathbb{R}^n. \text{ This implies } u(t,x) - \rho(x) \leq tL(0). \tag{4.2.1}$$

But, $u(t,x) = \min_{y\in\mathbb{R}^n} \left\{ \rho(y) + tL\left(\frac{x-y}{t}\right) \right\}$. Thus, we have

$$\begin{aligned}
u(t,x) &= \rho(x) + \min_{y\in\mathbb{R}^n} \left\{ \rho(y) - \rho(x) + tL\left(\frac{x-y}{t}\right) \right\} \\
&\geq \rho(x) + \min_{y\in\mathbb{R}^n} \left\{ -Lip(\rho)|x-y| + tL\left(\frac{x-y}{t}\right) \right\}, \quad \text{since } Lip(\rho) \geq \frac{|\rho(x)-\rho(y)}{|x-y|}| \\
&= \rho(x) - t \max_{z\in\mathbb{R}^n} \{Lip(\rho)|z| - L(z)\}, \quad \text{putting } z = \frac{x-y}{t} \\
&= \rho(x) - t \max_{v\in\bar{B}(0,Lip(\rho))} \left\{ \max_{z\in\mathbb{R}^n} \{w.z - L(z)\} \right\}, \text{since } \rho \text{ is continuous on } \bar{B}(0,Lip(\rho)) \\
&= \rho(x) - t \max_{v\in\bar{B}(0,Lip(\rho))} H(v), \quad \text{by assumption (4.1.1) and by definition of } H(v).
\end{aligned}$$

So,

$$\rho(x) - u(t,x) \leq t \max_{v\in\bar{B}(0,Lip(\rho)} H(v). \tag{4.2.2}$$

Combining (4.2.1) and (4.2.2) , we have $|u(t,x) - \rho(x)| \leq Ct$, where C given by (4.1.16). Thus, when t tends to 0, we get $0 \leq |u(0,x) - \rho(x)| \leq 0$. Hence, $u(0,x) = \rho(x)$.

ii) Let u is differentiable at $(t,x) \in (0,\infty) \times \mathbb{R}^n$. Take $h > 0$, $q \in \mathbb{R}^n$ fixed. Lemma (4.1.4) provides

$$\begin{aligned}
u(t+h, x+hq) &= \min_y \left\{ u(t,y) + hL\left(\frac{x+hq-y}{h}\right) \right\}, \quad \text{for small enough } h \\
&\leq u(t,x) + hL(q), \quad \text{putting } y = x.
\end{aligned}$$

This implies $\dfrac{u(t+h,x+hq) - u(t,x)}{h} \leq L(q), \quad \forall \ q \in \mathbb{R}^n.$

Subtracting and adding the same term $u(t+h,x)$ in the numerator yields

$$\frac{u(t+h,x+hq) - u(t+h,x)}{h} + \frac{u(t+h,x) - u(t,x)}{h} \leq L(q).$$

Letting $h \to 0$, using the definition of partial derivatives and differentiability of u, gives

$$q \cdot \nabla u(t,x) + u_t(t,x) \leq L(q).$$

Consequently

$$-u_t \geq \max_{q\in\mathbb{R}^n} \{q.\nabla u - L(q)\} = H(\nabla u) \quad \text{since } L^* = H.$$

Thus,

$$u_t(t,x) + H(\nabla u(t,x)) \leq 0. \tag{4.2.3}$$

Next to show the reverse inequality, take $z \in \mathbb{R}^n$ such that

$$u(t,x) = \rho(z) + tL\left(\frac{x-z}{t}\right), \tag{4.2.4}$$

which comes from Hopf-lax formula. We fix $h > 0$ and set

$$s = t - h, \; y = \frac{s}{t}x + (1 - \frac{s}{t})z. \tag{4.2.5}$$

Then

$$\frac{y - z}{s} = \frac{x - z}{t}. \tag{4.2.6}$$

Hence, we have

$$u(t,x) - u(s,y) \geq \rho(z) + tL\left(\frac{x-z}{t}\right) - \left[\rho(z) + sL\left(\frac{y-z}{s}\right)\right] = (t-s)L\left(\frac{x-z}{t}\right).$$

So, $$\frac{u(t,x) - u(s,y)}{t-s} \geq L\left(\frac{x-z}{t}\right).$$

This gives $$\frac{u(t,x) - u(t-h, \frac{s}{t}x + (1-\frac{s}{t})z)}{h} \geq L\left(\frac{x-z}{t}\right), \quad \text{by (4.2.5).}$$

That is, $$\frac{u(t,x) - u(t-h, (1-\frac{h}{t})x + \frac{h}{t}z}{h} \geq L\left(\frac{x-z}{t}\right).$$

Putting $q = \dfrac{x-z}{t}$ and then letting $h \to 0^+$, we get $q.\nabla u(t,x) + u_t(t,x) \geq L(q)$. Thus,

$$\begin{aligned} u_t(t,x) &\geq -(q.\nabla u(t,x) - L(q)) && (4.2.7)\\ &\geq -\max_{q\in\mathbb{R}^n}(q.\nabla u(t,x) - L(q)) = -H(\nabla u(t,x)). && (4.2.8)\end{aligned}$$

$$\text{Hence,} \quad u_t(t,x) + H(\nabla u(t,x)) \geq 0. \tag{4.2.9}$$

Therefore, combining (4.2.3) and (4.2.9), we conclude that

$$u_t(t,x) + H(\nabla u(t,x)) = 0.$$

□

4.2.2 Remark. From Corollary (4.1.6) and theorem (4.2.1), we conclude that the function u given by the Hopf-Lax formula is differentiable a.e. in $(0,\infty) \times \mathbb{R}^n$ and solves the IVP-HJE (4.1.1).

4.2.3 Theorem (Hopf-Lax formula as a viscosity solution)**.** *[Eva10]*

If the assumptions in remark (4.1.1) hold and ρ is bounded, then the function u given by the Hopf-Lax formula in equation (4.1.6) is a unique viscosity solution of the IVP-HJE

$$\begin{cases} u_t(t,x) + H(\nabla u(t,x)) = 0 & \text{in } (0,T] \times \mathbb{R}^n \\ u(0,x) = \rho(x), & (x \in \mathbb{R}^n). \end{cases}$$

Proof. This theorem is an extension of the uniqueness theorem (2.3.3). The proof comes from [Eva10] with some modification which I learned from [Hoa13].

1) Clearly u is Lipschitz continuous, differentiable a.e., solves the IVP-HJE on $(0,T] \times \mathbb{R}^n$ and $u(0,x) = \rho(x)$. This comes from lemma (4.1.5), corollary (4.1.6), theorem (4.2.1) and since $(0,T] \times \mathbb{R}^n \subseteq (0,T\infty) \times \mathbb{R}^n$. Also, since ρ is bounded, u is bounded $(0,T] \times \mathbb{R}^n$.

2) Let $v \in C^\infty((0,T)\times\mathbb{R}^n)$ and assume that $u-v$ has local minimum at $(t_0,x_0)\in(0,T)\times\mathbb{R}^n$. We have to show that

$$v_t(t_0,x_0)+H(\nabla v(t_0,x_0))\geq 0 \tag{4.2.10}$$

Let's assume the contrary. Assume that (4.2.10) is not true.

That is, $v_t(t_0,x_0)+H(\nabla v(t_0,x_0))<0$. Then there exists $\beta>0$, for (t,x) sufficiently close to (t_0,x_0) such that

$$v_t(t,x)+H(\nabla v(t,x))\leq -\beta<0. \tag{4.2.11}$$

Using the convex duality [Eva10, page 122] of H and L, we recall that

$$H(p)=\sup_{q\in\mathbb{R}^n}\{p.q-L(q)\}. \tag{4.2.12}$$

So, equation(4.2.11) becomes $v_t(t,x)+\sup_{q\in\mathbb{R}^n}\{q.\nabla v(t,x)-L(q)\}\leq-\beta$.

$$\Longrightarrow v_t(t,x)+q.\nabla v(t,x)\leq L(q)-\beta. \tag{4.2.13}$$

In view of lemma (4.1.4) replacing (t,x) by (t_0,x_0) and (s,y) by (t,x), we get

$$u(t_0,x_0)=\min_{x\in\mathbb{R}^n}\left\{u(t,x)+(t_0-t)L\left(\frac{x_0-x}{t_0-t}\right)\right\},\quad \text{for each}\quad 0\leq t\leq t_0. \tag{4.2.14}$$

Now for (t,x) sufficiently close to (t_0,x_0), with $(t,x)\neq(t_0,x_0)$, let's put $h=t_0-t$ and $hq=x_0-x$, for $0<t<t_0$. For some point x_1 close to x_0, with $x_1\neq x_0$, (4.2.14) yields

$$u(t_0,x_0)-u(t_0-h,x_1)=hL\left(\frac{x_0-x_1}{h}\right). \tag{4.2.15}$$

Now for sufficiently small $h>0$, we can compute

$$\begin{aligned}
v(t_0,x_0)-v(t_0-h,x_1) &= \int_0^1 \frac{d}{ds}v(sx_0+(1-s)x_1,t_0+(s-1)h)ds\\
&= \int_0^1 \nabla v(sx_0+(1-s)x_1,t_0+(s-1)h).(x_0-x_1)+\\
&+ v_t(sx_0+(1-s)x_1,t_0+(s-1)h)hds\\
&= h\int_0^1 \underbrace{\nabla v(...).q+v_t(...)}_{\leq L(q)-\beta}ds,\quad \text{where } q=\frac{x_0-x_1}{h}\quad \text{and by (4.2.13)}\\
&\leq h\int_0^1\left(L\left(\frac{x_0-x_1}{h}\right)-\beta\right)ds\\
&= hL\left(\frac{x_0-x_1}{h}\right)-\beta h\\
&= u(t_0,x_0)-u(x_1,t_0-h)-\beta h,\qquad \text{by (4.2.15)}.
\end{aligned}$$

This implies $(u-v)(t_0,x_0)\geq(u-v)(t_0-h,x_1)+\beta h$, for (t_0-h,x_1) sufficiently close to (t_0,x_0), but $(t_0-h,x_1)\neq(t_0,x_0)$. This is a contradiction to our hypothesis that $u-v$ has local minimum at (t_0,x_0). This contradiction comes from our wrong assumption that (4.2.10) is not true. Hence,

$$v_t(t_0,x_0)+H(\nabla v(t_0,x_0))\geq 0,\quad \text{as required.} \tag{4.2.16}$$

3) Next let $v \in C^\infty((0,T)\times\mathbb{R}^n)$ and suppose that

$$u - v \text{ has a local maximum at } (t_0, x_0) \text{ in } (0,T)\times\mathbb{R}^n. \tag{4.2.17}$$

We want to show that

$$v_t(t_0,x_0) + H(\nabla v(t_0,x_0)) \le 0.$$

Equation (4.2.14) implies that

$$u(t_0,x_0) - u(t,x) \le (t_0 - t)L\left(\frac{x_0 - x}{t_0 - t}\right), \quad 0 \le t \le t_0,\ x \in \mathbb{R}^n. \tag{4.2.18}$$

Then for (t,x) sufficiently close to (t_0,x_0), with $(t,x) \ne (t_0,x_0)$ equation (4.2.17) yields $(u-v)(t_0,x_0) \ge (u-v)(t,x)$. That is,

$$v(t_0,x_0) - v(t,x) \le u(t_0,x_0) - u(t,x). \tag{4.2.19}$$

Combining (4.2.19) with (4.2.18), one obtains

$$v(t_0,x_0) - v(t,x) \le (t_0 - t)L\left(\frac{x_0 - x}{t_0 - t}\right). \tag{4.2.20}$$

For (t,x) sufficiently close to (t_0,x_0), with $(t,x) \ne (t_0,x_0)$, set $h = t_0 - t$ and $hq = x_0 - x$. Then, (4.2.20) becomes

$$\frac{v(t_0,x_0) - v(t_0-h, x_0-hq)}{h} \le L(q).$$

But,

$$\begin{aligned}\frac{v(t_0,x_0) - v(t_0-h, x_0-hq)}{h} &= \frac{v(t_0,x_0) - v(t_0-h, x_0)}{h} + \frac{v(t_0-h,x_0) - v(t_0-h, x_0-hq)}{h}\\ &= \frac{v(t_0,x_0) - v(t_0-h, x_0)}{h} + q\cdot\frac{v(t_0,x_0) - v(t_0-h, x_0-hq)}{x_0 - x}.\end{aligned}$$

Rearranging and letting $h \to 0$, we have

$$v_t(t_0,x_0) + q.\nabla v(t_0,x_0) - L(q) \le 0,$$

which implies

$$\begin{aligned}-v_t(t_0,x_0) &\ge q.\nabla v(t_0,x_0) - L(q)\\ &\ge \sup_{q\in\mathbb{R}^n}(q.\nabla v(t_0,x_0) - L(q)) = H(\nabla v(t_0,x_0)), \text{ since } H^* = L,\ L^* = H.\end{aligned}$$

Hence, $v_t(t_0,x_0) + H(\nabla v(t_0,x_0)) \le 0$, as required.

Therefore, the function u defined by the Hopf-Lax formula (4.1.6) is a viscosity solution of the IVP-HJE (4.1.1) and the uniqueness comes from section (2.3) theorem (2.3.3) (uniqueness of viscosity solution).

□

4.2.4 Example. Using the Hopf-Lax formula, one can compute that

$$u(t,x) = -|x| - \frac{t}{2}, \quad (x \in \mathbb{R}^n,\ t \ge 0),$$

is a unique viscosity solution of the IVP for HJE

$$\begin{cases} u_t(t,x) + \frac{1}{2}|\nabla u(t,x)|^2 = 0, & \text{in } (0,T]\times\mathbb{R}^n\\ u(0,x) = -|x|, & x \in \mathbb{R}^n.\end{cases}$$

5. Conclusion

The main purpose of this essay was the study of Some general properties of viscosity solutions to Hamilton-Jacobi Equations (HJE) and we extended the study to Hamilton-Jacobi-Bellman Equation (HJBE) by introducing the notion of optimal control theory. Because of their non-linearity, in general case, the HJE and HJBE do not have classical solution, but viscosity solutions are a notion of weak solutions for a class of PDE of the HJ-type which are consistent in the classical sense.

Some of the main properties of a notion of viscosity solutions are:

• consistency with classical solutions.

• General existence of the viscosity solution can be deduced from approximation arguments using the vanishing viscosity method, by means of formulas such as the Value function and Hopf-Lax formula.

• An efficient way to prove their uniqueness theorem.

• A viscosity solution is stable with respect to uniform convergence. The importance of viscosity solution comes from its stability and consistency with the classical solution.

With the help of doubling the number of variables, we were able to prove the uniqueness of viscosity solutions and we presented two nice results, the value function and the Hopf-Lax formula, from optimal control theory where their back ground comes from the calculus of variations. We also proved that the value function and the Hopf-Lax formula provide a unique viscosity solution.

As a summary of section (3.2), the result we obtained represents a necessary condition for optimality. That is, if there is a control that minimizes the cost function, then this minimal cost satisfies the HJBE. Solving the HJBE is a sufficient condition for optimality [Eva10], [Bre01]. That is, a cost function that satisfies the HJBE is minimum cost functional. There fore, the HJBE is a beauty type of HJE which is a necessary and sufficient condition for optimality.

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It is time for Africa! Wake up!

References

[19777] *Hamilton-Jacobi Equation: A global approach: A global approach*, Mathematics in Science and Engineering, Elsevier Science, 1977.

[Ali11] D.S. Aliyu, *Nonlinear H-infinity control, Hamiltonian systems and Hamilton-Jacobi equations*, Taylor & Francis, 2011.

[Arn89] V.I. Arnol'd, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics, Springer, 1989.

[B+83] Stanley H Benton et al., *Review: Pl lions, generalized solutions of Hamilton-Jacobi equations*, Bulletin (New Series) of the American Mathematical Society **9** (1983), no. 2, 252–256.

[Bre01] Alberto Bressan, *Viscosity solutions of Hamilton-Jacobi equations and optimal control problems*, Lecture notes SISSA (2001).

[CEL84] Michael G Crandall, Lawrence C Evans, and P-L Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Transactions of the American Mathematical Society **282** (1984), no. 2, 487–502.

[CL83] Michael G Crandall and Pierre-Louis Lions, *Viscosity solutions of hamilton-jacobi equations*, Transactions of the American Mathematical Society **277** (1983), no. 1, 1–42.

[CP+11] Fabio Camilli, Emmanuel Prados, et al., *Viscosity solution*, Encyclopedia of Computer Vision (2011).

[CS04] P. Cannarsa and C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi Equations, and Optimal Control*, Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser Boston, 2004.

[Eva10] L.C. Evans, *Partial differential equations*, Graduate studies in mathematics, American Mathematical Society, 2010.

[GCIL87] Michael G. CRANDALL, Hitoshi ISHII, and Pierre-Louis LIONS, *Uniqueness of viscosity solutions of Hamilton-Jacobi equations revisited*, Journal of the Mathematical Society of Japan **39** (1987), no. 4, 581–596.

[Gom09] D. Gomes, *Viscosity solutions of Hamilton-Jacobi Equations*, Publicacoes matematicas, IMPA, 2009.

[Gre03] W. Greiner, *Classical mechanics: Systems of Particles and Hamiltonian Dynamics*, Classical theoretical physics, no. v. 1, Springer, 2003.

[Hoa13] N. Hoang, *Hopf-Lax formula and generalized characteristics*, ArXiv e-prints, arXiv: 1309.2547v2, math.AP (2013).

[Sho11] R.E. Showalter, *Hilbert space methods in partial differential equations*, Dover books on mathematics, Dover Publications, 2011.

[Van04] Tran Duc Van, *Hopf-Lax-Oleinik-type formulas for viscosity solutions to some Hamilton-Jacobi Equations*, Vietnam Journal of Mathematics **32** (2004), no. 3, 241–275.