Coincidence of Induced and Rigged Connections on Lightlike Hypersurfaces of Semi-Riemannian Manifolds

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Abstract

In this essay, we are concerned with the coincidence of induced and rigged connections on lightlike hypersurfaces of semi-Riemannian manifolds. We study for a given non vanishing function α on M the so-called α-associated metric $g_\alpha = g + \alpha\eta \otimes \eta$. We start by giving some fundamental equations for the case of semi-Riemannian hypersurfaces and later we provide similar (but very different) equations for lightlike hypersurfaces. We observe that in semi-Riemannian hypersurfaces, the induced connection always coincides with the Levi-Civita connection which is not always the case for lightlike hypersurfaces. We show that in the case of $\alpha = 1$, the coincidence holds when the second fundamental form B of TM, and the second fundamental form C of the screen distribution are equal and the rotation form is identically null. In the case where α is a non-vanishing function, the coincidence holds if the screen distribution is conformal with α as the conformal function and satisfying a certain relation.

French version

Dans ce mémoire, nous nous intéressons à la coïncidence entre connexion induite et connexion de Levi-Civita de la métrique associée sur des hypersurfaces de type lumière des variétés semi-riemanniennes. Nous commencerons par donner quelques équations fondamentales pour le cas des hypersurfaces semi-riemanniennes et plus tard nous fournissons des équations similaires (mais très différentes) pour les hypersurfaces de type lumière, nous observerons que dans les hypersurfaces semi-riemanniennes, la connexion induite coïncide toujours avec la connexion de Levi-Civita, ce qui n'est pas toujours le cas des hypersurfaces de type lumière. Nous verrons que dans le cas de $\alpha = 1$, la coïncidence est vérifiée lorsque la seconde forme fondamentale de TM, B et la seconde forme fondamentale de la distribution écran C sont égales et que la forme de rotation est identiquement nulle. Tandis que dans le cas où α est une fonction non nulle, la coïncidence est verifiée lorsque l'écran est conforme avec α comme fonction conforme et α satisfaisant une certaine relation.

Keywords: semi-Riemannian manifold, lightlike hypersurface, induced connexion, rigged connection.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Maryse Manuella Moutamal, 18 May 2018.

Contents

1. Introduction

A semi-Riemannian manifold $(\overline{M}, \overline{g})$ is a manifold endowed with a non-degenerate metric $\overline{g}$, precisely for any $x \in \overline{M}$, $\overline{g}_x$ is a non-degenerate bilinear form on $T_x\overline{M}$. A null hypersurface or lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is a co-dimension one submanifold M of $(\overline{M}, \overline{g})$ such that the restriction g of $\overline{g}$ on M is degenerate, i.e. $\forall x \in M$, the matrix of g_x on T_xM is a non invertible matrix. A rigged null hypersurface M is a one endowed with a rigging vector field. The latter being a vector field N on M such that $\forall x \in M, N_x$ is not tangent to M. The projection on M of the Levi-Civita connection $\overline{\nabla}$ of $\overline{M}$ along a null rigging N gives a so-called induced connection ∇ on M and this connection depends on the chosen rigging. If we set η to be the 1-form defined on M by $\eta(X) = g(N, X)$ then it is proved that for a given nowhere vanishing smooth function α on M, the following is a semi-Riemannian metric on M: $g_\alpha = g + \alpha\eta \otimes \eta$. This is called the α-associated metric. When $\alpha = 1$ the metric $g_1 = g + \eta \otimes \eta$ is usually called the induced metric or the rigged metric on M. In this work we study the conditions under which the Levi-Civita connection of the α-associated metric coincides with the induced connection. This essay is organized as it follows: this introduction is labeled chapter 1, the second chapter is devoted to the preliminaries of semi-Riemannian submanifolds, the third one is devoted to the lightlike hypersurfaces of semi-Riemannian manifolds, the fourth one studies the necessary conditions for coincidence between the induced and the rigged connections, and our last one is the conclusion.

2. Preliminaries on Semi-Riemannian Submanifolds

2.0.1 Definition (Manifold)**.** A manifold $\overline{M}$ of dimension n is an Hausdorff topological space, such that $\forall x \in \overline{M}$, $\exists\, U_x$ an open neighborhood of x in $\overline{M}$ and an homeomorphism $\varphi_x : U_x \to \mathbb{R}^n$, such that $\phi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a smooth function where (U_i, φ_i) is called a chart of M. We say that locally, M resembles to $\mathbb{R}^n$.

2.0.2 Definition (Submanifold)**.** Let M and $\overline{M}$ be two manifolds. M is said to be submanifold of $\overline{M}$ if $\exists\, i : M \to \overline{M}$ such that i is an embedding, i.e i is an immersion and an homeomorphism on its image. The integer $n - dim\, M$ is called the codimension of M in $\overline{M}$.

2.0.3 Definition. A submanifold of codimension 1 is called an **hypersurface**.

2.0.4 Proposition. Let M be a subset of $\mathbb{R}^n$. These 3 assertions are equivalent:

1. M is a submanifold of dimension m of $\mathbb{R}^n$;
2. $\forall x \in M$, $\exists\, U_x$ a neighborhood of x in M and $f : U_x \to \mathbb{R}^{n-m}$ a submersion such that $f^{-1}\{0\} = U_x \cap M$;
3. $\forall x \in M$, $\exists\, U_x$ a neighborhood of x in M, $\exists\, \Omega$ an open set of $\mathbb{R}^m$ and $h : \Omega \to \mathbb{R}^n$ an embedding such that $h(\Omega) = U_x \cap M$.

2.0.5 Definition. Let $\overline{M}$ be a manifold, a semi-Riemannian metric $\overline{g}$ over $\overline{M}$ is a 2-tensor covariant (i.e of type (0,2)), symmetric and non degenerate ($\ker(g) = \{0\}$). In other words $\forall x \in \overline{M}$ the linear form $\overline{g}_x : T_x\overline{M} \times T_x\overline{M} \to \mathbb{R}$ is a semi-Euclidian metric over the vector space $T_x\overline{M}$ and the application $x \mapsto \overline{g}_x$ is C^∞.

When $\overline{g}$ is a semi-Riemannian metric, the couple $(\overline{M}, \overline{g})$ is called a semi-Riemannian manifold .

2.0.6 Definition. Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold and M a submanifold of $\overline{M}$. We say that (M, g) is an isometrically immersed submanifold if $g = i^*\overline{g}$.

2.0.7 Definition (Semi-Riemannian submanifold)**.** Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold, and (M, g) an isometrically immersed submanifold of $(\overline{M}, \overline{g})$, M is said to be a semi-Riemannian submanifold of $\overline{M}$ if g is a semi-Riemannian metric.

2.0.8 Definition (Lightlike submanifold)**.** Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold, and (M, g) an immersed submanifold of $(\overline{M}, \overline{g})$, M is said to be a Lightlike submanifold of $\overline{M}$ if the restriction of $\overline{g}$ on M is degenerate i.e there exists a nonzero section X of TM such that $\overline{g}(X, Y) = 0,\ \forall Y \in \Gamma(TM)$.

2.0.9 Definition. Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold, a vector field X is said to be:

• spacelike if $\overline{g}(X, X) > 0$ or $X = 0$;

• timelike if $\overline{g}(X, X) < 0$;

• lightlike if $\overline{g}(X, X) = 0$ and $X \neq 0$.

2.0.10 Definition. The **index** of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is the number of negative numbers in a diagonal matrix associated to $\overline{g}$.

The index is also equal to the number of timelike vector contained in an orthonormal basis of $T\overline{M}$. Throughout, $(\overline{M}, \overline{g})$ is a semi-Riemannian manifold of index $q > 0$ and dimension n, and (M, g) is an immersed submanifold of $(\overline{M}, \overline{g})$.

2.1 Some Particular Distributions

2.1.1 Definition (Normal Distribution)**.** Let TM be the tangent bundle of M, we define the normal distribution $TM^{\perp}$ as the collection of normal subspaces $T_xM^{\perp}$ of $T_x\overline{M}$ defined by:

$$T_xM^{\perp} = \{X_x \in T_x\overline{M};\ \overline{g}(X_x, Y_x) = 0\ \ \forall Y_x \in T_xM\} \qquad \forall x \in M. \tag{2.1.1}$$

2.1.2 Definition (Radical distribution)**.** The radical distribution $Rad(TM)$ of TM is the collection of $Rad(T_xM)$ defined by:

$$Rad(T_xM) = \{X_x \in T_xM; \overline{g}(X_x, Y_x) = 0\ \ \forall Y_x \in T_xM\} \qquad \forall x \in M. \tag{2.1.2}$$

2.1.3 Remark.

- Observe that $Rad(TM) = TM \cap TM^{\perp}$
- $Rad(T_xM) = \{0\} \Leftrightarrow (M, g)$ is a semi-Riemannian submanifold of $(\overline{M}, \overline{g})$.

2.1.4 Proposition. From the above remark, it follows that (M, g) is a lightlike submanifold of $(\overline{M}, \overline{g})$ if and only if $Rad(T_xM) \neq \{0\}$.

2.2 Semi-Riemannian Hypersurfaces

Let $(\overline{M}, \overline{g})$ be a semi-Riemann manifold and (M, g) a semi-Riemannian hypersurface of $(\overline{M}, \overline{g})$; in this case $dim TM^{\perp} = 1$ and

$$T\overline{M}/_M = TM \perp TM^{\perp}. \tag{2.2.1}$$

The symbol $\perp$ simply means that $TM^{\perp}$ is orthogonal to TM and is the unique orthogonal complementary of TM.

2.2.1 Definition (Gauss map)**.** Let $x \in M$, there exists an open neighborhood U of x in M on which is defined a unique section of the normal bundle $TM^{\perp}$, denoted N such that $\overline{g}(N_y, N_y) = 1\ \forall y \in U$. $N : U \to TM^{\perp}$. N is called the Gauss map of M.

Let $\overline{\nabla}$ be the Levi-Civita connection of $(\overline{M}, \overline{g})$. From Equation (2.2.1), we can decompose $\overline{\nabla}_X Y$ as:

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N \qquad \text{(called Gauss formula)}, \tag{2.2.2}$$

where ∇ is called the induced connection and B the second fundamental form of M. It is easy to show that ∇ is a torsion-free connection on M, and B is a $2-$covariant symmetric tensor. Moreover ∇ is g-metric. Indeed, since $\overline{\nabla}$ is the Levi-Civita connection of $(\overline{M}, \overline{g})$, $\forall\ X, Y, Z \in \Gamma(TM)$,

$$\begin{aligned} X \cdot \overline{g}(Y, Z) &= \overline{g}(\overline{\nabla}_X Y, Z) + \overline{g}(\overline{\nabla}_X Z, Y) \\ \Longrightarrow\ X \cdot g(Y, Z) &= \overline{g}(\nabla_X Y + B(X, Y)N, Z) + \overline{g}(\nabla_X Z + B(X, Z)N, Y) \\ &= \overline{g}(\nabla_X Y, Z) + \overline{g}(\nabla_X Z, Y) \\ &= g(\nabla_X Y, Z) + g(\nabla_X Z, Y). \end{aligned}$$

Therefore ∇ is the Levi-Civita connection of (M, g).

2.2.2 Proposition. For every section X of TM, one has $\overline{\nabla}_X N \in \Gamma(TM)$.

Proof. $\overline{g}(N,N)=1 \implies X\cdot\overline{g}(N,N)=0 \implies 2\overline{g}(\overline{\nabla}_X N,N)=0 \implies \overline{\nabla}_X N\in\Gamma(TM)$.

□

The following formula is useful

$$\overline{\nabla}_X N = -A_N(X) \qquad \text{(called Weingarten formula)}, \tag{2.2.3}$$

where $A_N:\Gamma(TM)\to\Gamma(TM)$ is a field of endomorphisms called the shape operator of M.

2.2.3 Proposition. The shape operator A_N is g-self adjoint and related to the second fundamental form B by

$$B(X,Y)=g(A_N(X),Y), \qquad \forall\ X,Y\in\Gamma(TM). \tag{2.2.4}$$

Proof. $\overline{g}(Y,N)=0 \implies 0=X\cdot\overline{g}(Y,N)=\overline{g}(\overline{\nabla}_X Y,N)+\overline{g}(\overline{\nabla}_X N,Y)=B(X,Y)-\overline{g}(A_N(X),Y)$. □

2.2.4 Definition (geodesic)**.** Let (M,∇) be a manifold endowed with a linear connection. A curve $\gamma:I\to M$ is said to be a geodesic of M if

$$\nabla_{\gamma'}\gamma'=0. \tag{2.2.5}$$

Hence $\gamma:I\to M$ is a geodesic means that the velocity vector field γ' is parallely transported along the curve. Without any other precision in the case of semi-Riemannian Manifolds, the connection considered is the Levi-Civita connection and notions related to connection such as geodesic curve, are defined with respect the Levi-Civita connection.

2.2.5 Example. Geodesic curves for the real Euclidean space $\mathbb{R}^n$ are straight lines. Geodesic curves for the sphere S^n are great circles.

2.2.6 Definition (Totally geodesic)**.** A semi-Riemannian hypersurface M of $\overline{M}$ is said to be totally geodesic if its second fundamental form B vanishes.

2.2.7 Proposition ([6, page 104])**.** The following are equivalent:

1. M is totally geodesic in $\overline{M}$;
2. Every geodesic of M is also a geodesic of $\overline{M}$.

We can remark that the sphere S^n is not totally geodesic in the Euclidian space $\mathbb{R}^n$.

2.2.8 Definition (Totally umbilic)**.** A semi-Riemannian hypersurface M of $\overline{M}$ is said to be totally umbilic if there exists a smooth function ρ such that:

$$B=\rho\overline{g}$$

2.2.9 Proposition. For a semi-Riemannian hypersurface, we have the so-called Gauss-Codazzi equation:

$$\begin{aligned}\forall X,Y,Z,T\in\Gamma(TM),\quad \overline{g}(\overline{R}_{XY}Z,T) &= \overline{g}(R_{XY}Z,T)-B(X,T)B(Y,Z)+B(X,Z)B(Y,T)\\ \overline{g}(\overline{R}_{XY}Z,N) &= \overline{g}((\nabla_X A_N)Y-(\nabla_Y A_N)X,Z)\end{aligned}$$

where $\overline{R}$ and R are the Riemannian curvatures of $(\overline{M},\overline{g})$ and (M,g) respectively. And by definition $(\nabla_X A_N)Y=\nabla_X(A_N(Y))-A_N(\nabla_X Y)$.

Proof.

$$
\begin{aligned}
\overline{R}_{XY}Z &= \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z \\
&= \overline{\nabla}_X(\nabla_Y Z + B(Y,Z)N) - \overline{\nabla}_Y(\nabla_X Z + B(X,Z)N) - (\nabla_{[X,Y]}Z + B([X,Y],Z)N) \\
&= \overline{\nabla}_X\nabla_Y Z + \overline{\nabla}_X(B(Y,Z)N) - \overline{\nabla}_Y\nabla_X Z - \overline{\nabla}_Y(B(X,Z)N) - (\nabla_{[X,Y]}Z + B([X,Y],Z)N) \\
&= \nabla_X\nabla_Y Z + B(X,\nabla_Y Z)N + \overline{\nabla}_X(B(Y,Z)N) - \nabla_Y\nabla_X Z - B(Y,\nabla_X Z)N - \overline{\nabla}_Y(B(X,Z)N) \\
&\quad - (\nabla_{[X,Y]}Z + B([X,Y],Z)N) \\
&= R_{XY}Z + \overline{\nabla}_X(B(Y,Z)N) - \overline{\nabla}_Y(B(X,Z)N) + (B(X,\nabla_Y Z) \\
&\quad - B(Y,\nabla_X Z) - B([X,Y],Z))N \\
&= R_{XY}Z + (X\cdot B(Y,Z)N + B(X,Y)\overline{\nabla}_X N) - (Y\cdot B(X,Z)N + B(X,Z)\overline{\nabla}_Y N) \\
&\quad + (B(X,\nabla_Y Z) - B(Y,\nabla_X Z) - B([X,Y],Z))N \\
\overline{R}_{XY}Z &= R_{XY}Z + (X\cdot B(Y,Z)N - B(Y,Z)A_N(X)) - (Y\cdot B(X,Z)N - B(X,Z)A_N(Y)) \\
&\quad + (B(X,\nabla_Y Z) - B(Y,\nabla_X Z) - B([X,Y],Z))N. \qquad (2.2.6)
\end{aligned}
$$

If we take Equation (2.2.6) and we do the scalar product with T, we obtain:

$$\overline{g}(\overline{R}_{XY}Z,T) = \overline{g}(R_{XY}Z,T) - B(Y,Z)B(X,T) + B(X,Z)B(Y,T).$$

If we take Equation (2.2.6) and we do the scalar product with N, we obtain:

$$\overline{g}(\overline{R}_{XY}Z,N) = X\cdot B(Y,Z) - Y\cdot B(X,Z) + B(X,\nabla_Y Z) - B(Y,\nabla_X Z) - B([X,Y],Z).$$

Using Proposition 2.2.3,

$$
\begin{aligned}
\overline{g}(\overline{R}_{XY}Z,N) &= X\cdot\overline{g}(A_N(Y),Z) - Y\cdot\overline{g}(A_N(X),Z) + \overline{g}(A_N(X),\nabla_Y Z) \\
&\quad - \overline{g}(A_N(Y),\nabla_X Z) - \overline{g}(A_N([X,Y]),Z) \\
&= \overline{g}(\overline{\nabla}_X A_N(Y),Z) + \overline{g}(\overline{\nabla}_X Z, A_N(Y)) - \overline{g}(\overline{\nabla}_Y A_N(X),Z) - \overline{g}(\overline{\nabla}_Y Z, A_N(X)) \\
&\quad + \overline{g}(A_N(X),\nabla_Y Z) - \overline{g}(A_N(Y),\nabla_X Z) - \overline{g}(A_N(\nabla_X Y - \nabla_X Y),Z) \\
&= \overline{g}(\nabla_X A_N(Y),Z) - \overline{g}(\nabla_Y A_N(X),Z) - \overline{g}(A_N(\nabla_X Y - \nabla_X Y),Z) \\
&= \overline{g}((\nabla_X A_N)Y - (\nabla_Y A_N)X, Z).
\end{aligned}
$$

□

We have just studied some fundamental equations of the geometry of non degenerate sub-manifolds, in the next chapter we are going to study similar equations (but very different) for the case of lightlike hypersurfaces.

3. Lightlike Hypersurfaces of Semi-Riemannian Manifolds

In this chapter, (M, g) is a lightlike hypersurface of $(\overline{M}, \overline{g})$; this means that $Rad(TM) = TM \cap TM^{\perp} = TM^{\perp}$, therefore we can see that it is no more possible to decompose $T\overline{M}$ as in the case of a semi-Riemannian hypersurface; new tools are then needed.

3.0.1 Definition (Screen Distribution)**.** A screen distribution $S(TM)$ is a complementary distribution to $Rad(TM)$ in TM.

Let $S(TM)$ be a screen distribution for M. It is clear that $\dim S(TM) = n - 2$ and the following decomposition holds:

$$TM = Rad(TM) \oplus S(TM) \tag{3.0.1}$$

3.0.2 Proposition. The restriction of $\overline{g}$ on $S(TM)$ is non-degenerate.

Proof. Let $x \in M$ and $X_x \in S(T_xM)$ such that $\overline{g}_x(X_x, Y_x) = 0 \ \forall\, Y_x \in S(T_xM)$. Since $\overline{g}_x(X_x, Z_x) = 0, \ \forall\ Z_x \in Rad(T_xM)$ it follows from (3.0.1) that $\overline{g}_x(X_x, Y_x) = 0, \ \forall\ Y_x \in T_xM$. Hence $X_x \in Rad(T_xM) \cap S(T_xM) = \{0\}$ which implies that $X_x = 0$. Therefore $(S(TM), \overline{g})$ is non-degenerate. □

3.1 Decomposition of $T\overline{M}$

• Approach of Duggal and Bejancu

3.1.1 Theorem ([3, page 79])**.** *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying:*

$$\overline{g}(N, \xi) = 1 \tag{3.1.1}$$

and

$$\overline{g}(N, N) = \overline{g}(N, W) = 0, \quad \forall W \in \Gamma(S(TM)_{|\mathcal{U}}) \tag{3.1.2}$$

$$T\overline{M}|_M = TM \oplus tr(TM) \tag{3.1.3}$$

$tr(TM)$ is called the transversal bundle of TM.

• Rigging techniques

3.1.2 Definition (Rigging)**.** We call rigging for M a vector field ζ defined on an open set of $\overline{M}$ containing M such that

$$\zeta_x \notin T_xM, \ \forall\ x \in M$$

Let $\overline{\eta}$ be the 1-form metrically equivalent to ζ (i.e $\overline{\eta} = \overline{g}(\zeta, .)$) and $\tilde{g} = \overline{g} + \overline{\eta} \otimes \overline{\eta}$. $\overline{\eta}$ is a $1-$form over $\overline{M}$ and $\tilde{g}$ is a $2-$covariant tensor over $\overline{M}$ called the $1-$twisted metric or just the twisted metric on $\overline{M}$.

Let $g_\eta = i^*\tilde{g}$; $\eta = i^*\overline{\eta}$; η is a $1-$form over M and g_η is a $2-$covariant tensor over M called the rigged metric or the $1-$associated metric or just the associated metric on M.

$$g_\eta = g + \eta \otimes \eta \tag{3.1.4}$$

Let us set

$$S(TM) = \ker \eta \quad = \quad \{X \in TM, \eta(X) = 0\}$$

since ζ is not a section of TM, it follows that $\overline{g}(X, \zeta) = \eta(X) \neq 0,\ \forall X \in \Gamma(Rad(TM)) \setminus \{0\}$. This shows that $S(TM)$ is a screen distribution of TM.

$$TM = Rad(TM) \oplus_\perp S(TM) \tag{3.1.5}$$

3.1.3 Lemma. g_η is non-degenerate.

Proof. Let X be a nonzero section of TM such that $g_\eta(X, Y) = 0 \;\; \forall\, Y \in \Gamma(TM)$.

From the decomposition (3.1.5), there exists $X_1 \in Rad(TM),\ X_2 \in S(TM)$ such that $X = X_1 + X_2$,

$$\begin{aligned} 0 = g_\eta(X, Y) = g(X, Y) + \eta(X)\eta(Y) &= g(X_1, Y) + g(X_2, Y)) + \eta(X_1)\eta(Y)) + \eta(X_2)\eta(Y) \\ &= g(X_2, Y)) + \eta(X_1)\eta(Y). \end{aligned}$$

In particular for $X_1 \in Rad(TM) \subset TM$,

$0 = g(X_2, X_1) + \eta(X_1)^2 = \eta(X_1)^2$, so it follows that $X_1 \in S(TM) \cap Rad(TM) = \{0\}$, therefore $X = X_2$. And we have

$$0 = g_\eta(X_2, Y) = \overline{g}(X_2, Y) + \eta(X_2)\eta(Y) = \overline{g}(X_2, Y) = 0 \;\; \forall\, Y \in TM. \tag{3.1.6}$$

But we also have that $\overline{g}(X_2, \zeta) = \eta(X_2) = 0$. Since $T\overline{M} = TM \oplus \langle \zeta \rangle$, it follows that $\overline{g}(X_2, Y) = 0\ \forall Y \in\ \Gamma(T\overline{M})$ and the fact that $\overline{g}$ is non-degenerate leads to $X_2 = 0$, hence $X = 0$.

Therefore g_η is non degenerate. □

3.1.4 Definition (Rigged)**.** The vector field ξ over M metrically equivalent to η i.e ($\eta = g_\eta(\xi, .)$) is called the **rigged vector field** associated to ζ.

3.1.5 Lemma. The rigged vector field ξ is the unique section of the radical distribution such that $\overline{g}(\xi, \zeta) = 1$.

Proof. First of all, since ζ is not a section of TM, it follows that

$$\overline{g}(X, \zeta) \neq 0, \qquad \forall X \in \Gamma(Rad(TM)) \setminus \{0\}. \tag{3.1.7}$$

We have $\eta(X) = g_\eta(\xi, X) = g(\xi, X) + \eta(X)\eta(\xi)$. Hence $\eta(X)(1 - \eta(\xi)) = g(\xi, X) \;\; \forall X \in \Gamma(TM)$ In particular for $X \in \Gamma(Rad(TM)) \setminus \{0\}$, $\overline{g}(X, \zeta)(1 - \overline{\eta}(\xi)) = 0$. Using (3.1.7) the latter leads to $1 - \eta(\xi) = 0$. This is

$$\eta(\xi) = \overline{g}(\xi, \zeta) = 1. \tag{3.1.8}$$

For every section X of TM, $\eta(X) = g(\xi, X) + \eta(X) \implies g(\xi, X) = 0$. Hence,

$$Rad(TM) = \langle \xi \rangle. \tag{3.1.9}$$

□

3.1.6 Remark. Locally, it is always possible to define a vector field which span the radical distribution; but ξ is globally defined.

The vector field $N = \zeta - \frac{1}{2}\overline{g}(\zeta,\zeta)\xi$ is a lightlike transversal vector field of M and it satisfies $\overline{g}(N,\xi) = 1$.

Proof.

$$\overline{g}(N,N) = \overline{g}\left(\zeta - \frac{1}{2}\overline{g}(\zeta,\zeta)\xi, \zeta - \frac{1}{2}\overline{g}(\zeta,\zeta)\xi\right) = \overline{g}(\zeta,\zeta) - \overline{g}(\zeta,\zeta)\overline{g}(\xi,\zeta) + \frac{1}{4}\overline{g}(\zeta,\zeta)\overline{g}(\xi,\xi) = 0$$
$$\overline{g}(N,\xi) = \overline{g}\left(\zeta - \frac{1}{2}\overline{g}(\zeta,\zeta)\xi, \xi\right) = \overline{g}(\zeta,\xi) - \frac{1}{2}\overline{g}(\zeta,\zeta)\overline{g}(\xi,\xi) = \overline{g}(\zeta,\xi) = 1.$$

□

3.1.7 Remark. For lightlike hypersurfaces, giving a rigging vector field, we can always have a null rigging vector field, so without lost of generality, we will always talk about null rigged hypersurfaces.

From Equation (3.1.3), We have the following decompositions. $\forall\ X, Y \in \Gamma(TM)$

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N \tag{3.1.10}$$
$$\overline{\nabla}_X N = \tau(X)N - A_N(X) \tag{3.1.11}$$

where A_N is called the shape operator of M and B is the the second fundamental form of M.

3.1.8 Proposition. $A_N(U) \in TM$, ∇ is a connection over TM, and B is a symmetric tensor.

Proof. We know that $\overline{\nabla}$ is a connection over $T\overline{M}$ so it satisfies

$$\overline{\nabla}_{fX} Y = f\overline{\nabla}_X Y$$
$$\overline{\nabla}_X fY = X.(f)Y + f\overline{\nabla}_X Y$$
$$\overline{\nabla}_X (Y+Z) = \overline{\nabla}_X Y + \overline{\nabla}_X Z$$
$$\overline{\nabla}_{X+Y} Z = \overline{\nabla}_X Z + \overline{\nabla}_Y Z$$

$$\begin{aligned}
\overline{\nabla}_{fX} Y &= f\overline{\nabla}_X Y \\
\Longrightarrow \nabla_{fX} Y + B(fX,Y)N &= f(\nabla_X Y + B(X,Y)N) \\
\Longrightarrow \nabla_{fX} Y - f\nabla_X Y &= (fB(X,Y) - B(fX,Y))N
\end{aligned}$$

We know that $TM \cap tr(TM) = \{0\}$

$$\begin{aligned}
\Longrightarrow \nabla_{fX} Y - f\nabla_X Y &= (fB(X,Y) - B(fX,Y))N = 0 \\
\Longrightarrow \nabla_{fX} Y = f\nabla_X Y \quad &\text{and} \quad fB(X,Y) = B(fX,Y)
\end{aligned}$$

$$\begin{aligned}
& \overline{\nabla}_X fY = X.(f)Y + f\overline{\nabla}_X Y \\
\Longrightarrow\ & \nabla_X fY + B(X,fY)N = X.(f)Y + f(\nabla_X Y + B(X,Y)N) \\
\Longrightarrow\ & \nabla_X fY - X.(f)Y - f\nabla_X Y = (fB(X,Y) - B(X,fY))N \\
TM \cap tr(TM) = \{0\} \quad \Longrightarrow\ & \nabla_X fY = X.(f)Y + f\nabla_X Y \text{ and } fB(X,Y) = B(X,fY)
\end{aligned}$$

$$
\begin{aligned}
& \overline{\nabla}_X(Y+Z) = \overline{\nabla}_X Y + \overline{\nabla}_X Z \\
\Longrightarrow\ & \nabla_X(Y+Z) + B(X,Y+Z)N = \nabla_X Y + B(X,Y)N + \nabla_X Z + B(X,Z)N \\
\Longrightarrow\ & \nabla_X(Y+Z) - \nabla_X Y - \nabla_X Z = (-B(X,Y+Z) + B(X,Y) + B(X,Z))N \\
TM \cap tr(TM) = \{0\} \Longrightarrow\ & \\
\nabla_X(Y+Z) =\ & \nabla_X Y + \nabla_X Z \text{ and } B(X,Y+Z) = B(X,Y) + B(X,Z)
\end{aligned}
$$

$$
\begin{aligned}
& \overline{\nabla}_{X+Y} Z = \overline{\nabla}_X Z + \overline{\nabla}_Y Z \\
\Longrightarrow\ & \nabla_{X+Y} Z + B(X+Y,Z)N = \nabla_X Z + B(X,Z)N + \nabla_Y Z + B(Y,Z)N \\
\Longrightarrow\ & \nabla_{X+Y} Z - \nabla_X Z - \nabla_Y Z = (-B(X+Y,Z) + B(X,Z) + B(Y,Z))N \\
TM \cap tr(TM) = \{0\} \Longrightarrow\ & \\
\nabla_{X+Y} Z =\ & \nabla_X Z + \nabla_Y Z \text{ and } B(X+Y,Z) = B(X,Z) + B(Y,Z)
\end{aligned}
$$

$$
\begin{aligned}
& \overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N \\
& \overline{\nabla}_Y X = \nabla_Y X + B(Y,X)N \\
& \overline{\nabla}_X Y - \overline{\nabla}_Y X = \nabla_X Y - \nabla_Y X + (B(X,Y) - B(Y,X))N \\
& \overline{\nabla} \text{ is the Levi-Civita connection of } \overline{M} \\
& [X,Y] = \nabla_X Y - \nabla_Y X + (B(X,Y) - B(Y,X))N \\
& [X,Y] - \nabla_X Y + \nabla_Y X = (B(X,Y) - B(Y,X))N \\
TM \cap tr(TM) = \{0\} \Longrightarrow\ & \nabla_X Y - \nabla_Y X - [X,Y] = 0 \text{ and } B(X,Y) = B(Y,X)
\end{aligned}
$$

Therefore ∇ is a connection without torsion over TM, and B is a symmetric tensor over TM. □

3.1.9 Definition (Totally geodesic)**.** The lightlike hypersurface M is said to be totally geodesic if $\forall X, Y \in TM$,

$$B(X,Y) = 0. \tag{3.1.12}$$

3.1.10 Definition (Totally umbilical)**.** M is said to be totally umbilical if there exists a function $f \in C^\infty(M)$ such that

$$B(X,Y) = f\overline{g}(X,Y) \quad \forall X, Y \in TM. \tag{3.1.13}$$

3.1.11 Proposition. ∇ is a connection over M which is not $\overline{g}$-metric and it satisfies:

$$(\nabla_X \overline{g})(Y,Z) = B(X,Y)\overline{g}(N,Z) + B(X,Z)\overline{g}(N,Y) \quad \forall X,Y,Z \in \Gamma(TM) \tag{3.1.14}$$

$$B(X,Y) = -\overline{g}(\nabla_X \xi, Y) \tag{3.1.15}$$

Proof.

$$
\begin{aligned}
\overline{\nabla}_X Y &= \nabla_X Y + B(X,Y)N \\
\overline{g}(\overline{\nabla}_X Y, Z) &= \overline{g}(\nabla_X Y, Z) + \overline{g}(B(X,Y)N, Z) \\
\overline{g}(\overline{\nabla}_X Z, Y) &= \overline{g}(\nabla_X Z, Y) + \overline{g}(B(X,Z)N, Y)
\end{aligned}
$$

since $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$, then

$$
\begin{aligned}
X.\overline{g}(Y,Z) &= \overline{g}(\overline{\nabla}_X Y, Z) + \overline{g}(Y, \overline{\nabla}_X Z) \\
&= \overline{g}(\nabla_X Y, Z) + \overline{g}(B(X,Y)N, Z) + \overline{g}(\nabla_X Z, Y) + \overline{g}(B(X,Z)N, Y)
\end{aligned}
$$

$$\begin{aligned} X.\overline{g}(Y,Z) - \overline{g}(\nabla_X Y, Z) - \overline{g}(Y, \nabla_X Z) &= \overline{g}(N,Z)B(X,Y) + \overline{g}(N,Y)B(X,Z) \\ (\nabla_X \overline{g})(Y,Z) &= \overline{g}(N,Z)B(X,Y) + \overline{g}(N,Y)B(X,Z). \end{aligned}$$

From Equation (3.1.10), we have

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + B(X,Y)N \\ \overline{g}(\overline{\nabla}_X Y, \xi) &= \overline{g}(\nabla_X Y, \xi + B(X,Y)\overline{g}(N,\xi) \\ X.\overline{g}(Y,\xi) - \overline{g}(\nabla_X \xi, Y) &= \overline{g}(\nabla_X Y, \xi) + B(X,Y) \\ -\overline{g}(\nabla_X \xi, Y) &= B(X,Y). \end{aligned}$$

□

3.1.12 Proposition. The second fundamental form B satisfies $B(\xi, .) = 0$.

Proof. From (3.1.14), if we replace Y and Z by ξ, we obtain:

$$\begin{aligned} \overline{g}(N,\xi)B(X,\xi) + \overline{g}(N,\xi)B(X,\xi) &= (\nabla_X \overline{g})(\xi,\xi) \\ 2B(X,\xi) &= X.\overline{g}(\xi,\xi) - 2\overline{g}(\nabla_X \xi, \xi) = 0. \end{aligned}$$

□

Since $B(\xi, .) = 0$, from Equation (3.1.10), we have:

$$\overline{\nabla}_X \xi = \nabla_X \xi \in TM.$$

So using the decomposition (3.1.5), we can decomposed it as follow $\nabla_X \xi = a\xi - \overset{\star}{A}_\xi(X)$ with $\overset{\star}{A}_\xi(X) \in S(TM)$ and $a = \overline{g}(\nabla_X \xi, N)$ and it is easy to verify that $a = -\tau(X)$.

$$\nabla_X \xi = -\tau(X)\xi - \overset{\star}{A}_\xi(X) \quad . \tag{3.1.16}$$

where $\overset{\star}{A}_\xi$ is a field endomorphism called the shape operator of S(TM).

3.1.13 Proposition. The shape operator $\overset{\star}{A}_\xi$ is g-self-adjoint endomorphism and related to the second fundamental form B by: $\forall X, Y, Z \in TM$,

$$B(X,Y) = \overline{g}(\overset{\star}{A}_\xi(X), Y)$$

Proof. From Equation (3.1.16),

$$\begin{aligned} \nabla_X \xi &= -\tau(X)\xi - \overset{\star}{A}_\xi(X) \\ \overline{g}(\nabla_X \xi, Y) &= \overline{g}(-\tau(X)\xi - \overset{\star}{A}_\xi(X), Y) = -\tau(X)\overline{g}(\xi, Y) - \overline{g}(\overset{\star}{A}_\xi(X), Y) \\ \text{from (3.1.15)} \quad -B(X,Y) = \overline{g}(\nabla_X \xi, Y) &= -\overline{g}(\overset{\star}{A}_\xi(X), Y) \\ B(X,Y) &= \overline{g}(\overset{\star}{A}_\xi(X), Y). \end{aligned}$$

□

$\forall X\ , Y \in \Gamma(TM)\ ,\ \ \nabla_X PY\ \ \in \Gamma(TM)$, we can decomposed it as follows:

$$\nabla_X PY \quad = \quad \nabla^*_X PY + C(X, PY)\xi. \tag{3.1.17}$$

Where $\nabla^*_X PY\ \ \in S(TM)$ and C is a 2-tensor called the second fundamental form associated to the screen distribution, and C satisfies:

$$C(X, PY) = -\overline{g}(\overline{\nabla}_X N, PY) \quad = \quad \overline{g}(A_N(X), Y) \tag{3.1.18}$$

$$C(PX, PY) - C(PY, PX) \quad = \quad \overline{g}(N, [PX, PY]). \tag{3.1.19}$$

Proof.

$$\begin{aligned} 0 &= \overline{g}(N, PY) \\ 0 &= X \cdot \overline{g}(N, PY) = \overline{g}(\overline{\nabla}_X N, PY) + \overline{g}(N, \overline{\nabla}_X PY) \\ &= -\overline{g}(A_N(X), PY) + \overline{g}(N, \nabla_X PY) \\ &= -\overline{g}(A_N(X), PY) + C(X, PY). \end{aligned}$$

And we have the result. From the previous result:

$$\begin{aligned} C(PX, PY) - C(PY, PX) &= \overline{g}(\overline{\nabla}_{PX} PY, N) - \overline{g}(\overline{\nabla}_{PY} PX, N) \\ &= \overline{g}([PX, PY], N) \end{aligned}$$

□

3.1.14 Lemma. If S(TM) is integrable, i.e $\forall\ X, Y \in \Gamma(TM),\ [PX, PY] \in S(TM)$, then C is symmetric in S(TM) and ∇^* is a torsion-free connection which is $\overline{g}$-metric.

Proof. From the previous result,

$$C(PX, PY) - C(PY, PX) \quad = \quad \overline{g}([PX, PY], N) = 0.$$

$$\begin{aligned} T_{\nabla^*}(PX, PY) &= \nabla^*_{PX} PY - \nabla^*_{PY} PX - [PX, PY] \\ &= \nabla_{PX} PY - C(PX, PY) - \nabla_{PY} PX + C(PY, PX) - [PX, PY] \\ &= \nabla_{PX} PY - \nabla_{PY} PX - [PX, PY] \\ &= T_{\nabla}(PX, PY) = 0 \end{aligned}$$

$$\begin{aligned} \nabla^*_{PX}\overline{g}(PY, PZ) &= PX \cdot \overline{g}(PY, PZ) - \overline{g}(\nabla^*_{PX} PY, PZ) - \overline{g}(\nabla^*_{PX} PZ, PY) \\ &= PX \cdot \overline{g}(PY, PZ) - \overline{g}(\nabla_{PX} PY - C(PX, PY)\xi, PZ) - \overline{g}(\nabla_{PX} PZ - C(PX, PZ)\xi, PY) \\ &= PX \cdot \overline{g}(PY, PZ) - \overline{g}(\nabla_{PX} PY, PZ) - \overline{g}(\nabla_{PX} PZ, PY) \\ &= PX \cdot \overline{g}(PY, PZ) - \overline{g}(\overline{\nabla}_{PX} PY, PZ) - \overline{g}(\overline{\nabla}_{PX} PZ, PY) \\ &= 0 \text{ Since } \overline{\nabla} \text{ is the levi-Civita of } \overline{g} \end{aligned}$$

□

3.1.15 Remark. Note that for the case of lightlike hypersurface, A_N is no more g-self adjoint like in the semi-Riemannian case, but it is g-self adjoint when the screen distribution is integrable.

The Riemannian curvature R of ∇ is given by

$$R_{XY}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \quad \forall X,Y,Z \in TM. \tag{3.1.20}$$

3.1.16 Proposition. The Riemannian curvature R of ∇ and The Riemannian curvature $\overline{R}$ of $\overline{\nabla}$ are related by:

$$R_{XY}\xi = \overline{R}_{XY}\xi. \tag{3.1.21}$$

Proof. We have seen that $\overline{\nabla}_X\xi = \nabla_X\xi = -\tau(X)\xi - \overset{\star}{A}_\xi(X)$,

and in Proposition 3.1.13, $B(X, \overset{\star}{A}_\xi(Y)) = B(\overset{\star}{A}_\xi(X), Y)$ so

$$\begin{aligned}
R_{XY}\xi &= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi \\
&= \nabla_X(-\tau(Y)\xi - \overset{\star}{A}_\xi(Y)) - \nabla_Y(-\tau(X)\xi - \overset{\star}{A}_\xi(X)) - \overline{\nabla}_{[X,Y]}\xi \\
&= -\overline{\nabla}_X\tau(Y)\xi - \nabla_X\overset{\star}{A}_\xi(Y) + \overline{\nabla}_Y\tau(X)\xi + \nabla_Y\overset{\star}{A}_\xi(X) - \overline{\nabla}_{[X,Y]}\xi \\
&= -\overline{\nabla}_X\tau(Y)\xi - (\overline{\nabla}_X\overset{\star}{A}_\xi(Y) - B(X, \overset{\star}{A}_\xi(Y))) + \overline{\nabla}_Y\tau(X)\xi + \overline{\nabla}_Y\overset{\star}{A}_\xi(X) - B(Y, \overset{\star}{A}_\xi(X)) - \overline{\nabla}_{[X,Y]}\xi \\
&= -\overline{\nabla}_X\tau(Y)\xi - \overline{\nabla}_X\overset{\star}{A}_\xi(Y) + \overline{\nabla}_Y\tau(X)\xi + \overline{\nabla}_Y\overset{\star}{A}_\xi(X) - \overline{\nabla}_{[X,Y]}\xi \\
&= \overline{\nabla}_X\overline{\nabla}_Y\xi - \overline{\nabla}_Y\overline{\nabla}_X\xi - \overline{\nabla}_{[X,Y]}\xi \\
&= \overline{R}_{XY}\xi.
\end{aligned}$$

□

3.1.17 Proposition. (**Gauss-Codazzi Equations**)

For a lightlike hypersurface, we have the so-called Gauss-Codazzi equations:

$\forall\ X, Y, Z, U \in \Gamma(TM)$,

$$\begin{aligned}
\overline{g}(\overline{R}_{XY}Z, PU) &= \overline{g}(R_{XY}Z, PU) + B(X,Z)\overline{g}(A_N(Y), PU) - B(Y,Z)\overline{g}(A_N(X), PU) \\
\overline{g}(\overline{R}_{XY}Z, \xi) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\
\overline{g}(\overline{R}_{XY}Z, N) &= \overline{g}(R_{XY}Z, N).
\end{aligned} \tag{3.1.22}$$

Proof.

$$\begin{aligned}
\overline{R}_{XY}Z &= \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z \\
&= \overline{\nabla}_X(\nabla_Y Z + B(Y,Z)N) - \overline{\nabla}_Y(\nabla_X Z + B(X,Z)N) - (\nabla_{[X,Y]}Z + B([X,Y],Z)N) \\
&= \overline{\nabla}_X\nabla_Y Z + \overline{\nabla}_X(B(Y,Z)N) - \overline{\nabla}_Y\nabla_X Z - \overline{\nabla}_Y(B(X,Z)N) - (\nabla_{[X,Y]}Z + B([X,Y],Z)N) \\
&= \nabla_X\nabla_Y Z + B(X,\nabla_Y Z)N + \overline{\nabla}_X(B(Y,Z)N) - \nabla_Y\nabla_X Z - B(Y,\nabla_X Z)N - \overline{\nabla}_Y(B(X,Z)N) \\
&\quad -(\nabla_{[X,Y]}Z + B([X,Y],Z)N) \\
&= R_{XY}Z + \overline{\nabla}_X(B(Y,Z)N) - \overline{\nabla}_Y(B(X,Z)N) + (B(X,\nabla_Y Z) \\
&\quad -B(Y,\nabla_X Z) - B([X,Y],Z))N \\
&= R_{XY}Z + (X\cdot B(Y,Z)N + B(X,Y)\overline{\nabla}_X N) - (Y\cdot B(X,Z)N + B(X,Z)\overline{\nabla}_Y N) \\
&\quad +(B(X,\nabla_Y Z) - B(Y,\nabla_X Z) - B([X,Y],Z))N \\
&= R_{XY}Z + (X\cdot B(Y,Z)N + B(Y,Z)(\tau(X)N - A_N(X)) - (Y\cdot B(X,Z)N + B(X,Z)(\tau(Y)N \\
&\quad -A_N(Y)) + (B(X,\nabla_Y Z) - B(Y,\nabla_X Z) - B([X,Y],Z))N
\end{aligned}$$

$$\begin{aligned} &= R_{XY}Z + [X\cdot B(Y,Z) - Y\cdot B(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) + B(X,\nabla_Y Z) \\ &\quad -B(Y,\nabla_X Z) - B([X,Y],Z)]N - A_N(X)B(Y,Z) + A_N(Y)B(X,Z). \end{aligned} \tag{3.1.23}$$

By doing the scalar product of Equation (3.1.23) with $PU \in S(TM)$ we obtain:

$$\begin{aligned} \overline{g}(\overline{R}_{XY}Z, PU) &= \overline{g}(R_{XY}Z, PU) + \overline{g}(-A_N(X)B(Y,Z) + A_N(Y)B(X,Z), PU) \\ &= \overline{g}(R_{XY}Z, PU) - B(Y,Z)\overline{g}(A_N(X), PU) + B(X,Z)\overline{g}(A_N(Y), PU). \end{aligned}$$

By doing the scalar product of Equation (3.1.23) with ξ we obtain:

$$\begin{aligned} \overline{g}(\overline{R}_{XY}Z,\xi) &= X\cdot B(Y,Z) - Y\cdot B(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ &\quad +B(X,\nabla_Y Z) - B(Y,\nabla_X Z) - B([X,Y],Z) \\ &= X\cdot B(Y,Z) - Y\cdot B(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ &\quad +B(X,\nabla_Y Z) - B(Y,\nabla_X Z) - B(\nabla_X Y - \nabla_Y X, Z) \\ &= X\cdot B(Y,Z) - B(Y,\nabla_X Z) - B(\nabla_X Y, Z) - Y\cdot B(X,Z) + B(X,\nabla_Y Z) + B(\nabla_Y X, Z) \\ &\quad +\tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z). \end{aligned}$$

By doing the scalar product of Equation (3.1.23) with N we obtain:

$$\begin{aligned} \overline{g}(\overline{R}_{XY}Z, N) &= \overline{g}(R_{XY}Z, N) + \overline{g}(-A_N(X)B(Y,Z) + A_N(Y)B(X,Z), N) \\ &= \overline{g}(R_{XY}Z, N) - B(Y,Z)\overline{g}(A_N(X), N) + B(X,Z)\overline{g}(A_N(Y), N) \\ &= \overline{g}(R_{XY}Z, N) - B(Y,Z)\overline{g}(\overline{\nabla}_X N, N) + B(X,Z)\overline{g}(\overline{\nabla}_X N, N) \\ \text{We know that} &\quad \overline{g}(N,N) = 0 \implies X\cdot\overline{g}(N,N) = 0 \implies \overline{g}(\overline{\nabla} X N, N) = 0 \\ \overline{g}(\overline{R}_{XY}Z, N) &= \overline{g}(R_{XY}Z, N). \end{aligned}$$

□

3.1.18 Proposition. From the Gauss-Codazzi equations, we deduce that:

$$\begin{aligned} \overline{g}(R_{XY}PZ, N) &= (\nabla^*_X C)(Y,PZ) - (\nabla^*_Y C)(X,PZ) + \tau(X)C(Y,PZ) - \tau(Y)C(X,PZ) \\ \overline{g}(R_{XY}\xi, N) &= C(Y, \overset{\star}{A}_\xi(X)) - C(X, \overset{\star}{A}_\xi(Y)) - d\tau(X,Y). \end{aligned} \tag{3.1.24}$$

Proof. In Equation (3.1.22), by replacing Z by PZ we have

$$\begin{aligned} R_{XY}PZ &= \nabla_X\nabla_Y PZ - \nabla_Y\nabla_X PZ - \nabla_{[X,Y]}PZ \\ &= \nabla_X(\nabla^*_Y PZ + C(Y,PZ)\xi) - \nabla_Y(\nabla^*_X PZ + C(X,PZ)\xi) - \nabla^*_{[X,Y]}PZ - C([X,Y],PZ)\xi \\ &= \nabla^*_X\nabla^*_Y PZ + C(X,\nabla^*_Y PZ)\xi + X\cdot C(Y,PZ)\xi + C(Y,PZ)\nabla_X\xi - \nabla^*_Y\nabla^*_X PZ \\ &\quad -C(Y,\nabla^*_X PZ)\xi - Y\cdot C(X,PZ)\xi - C(X,PZ)\nabla_Y\xi - \nabla^*_{[X,Y]}PZ - C([X,Y],PZ)\xi \\ &= \nabla^*_X\nabla^*_Y PZ + C(X,\nabla^*_Y PZ)\xi + X\cdot C(Y,PZ)\xi + C(Y,PZ)(-\tau(X)\xi - \overset{\star}{A}_\xi(X)) - \nabla^*_Y\nabla^*_X PZ \\ &\quad -C(Y,\nabla^*_X PZ)\xi - Y\cdot C(X,PZ)\xi - C(X,PZ)(-\tau(Y)\xi - \overset{\star}{A}_\xi(Y)) - \nabla^*_{[X,Y]}PZ - C([X,Y],PZ)\xi \end{aligned}$$

$$
\begin{aligned}
\overline{g}(\overline{R}_{XY}PZ, N) &= \overline{g}(R_{XY}PZ, N) \\
&= C(X, \nabla^*_Y PZ) + X \cdot C(Y, PZ) - C(Y, \nabla^*_X PZ) - Y \cdot C(X, PZ) - C([X, Y], PZ) \\
&\quad -C(Y, PZ)\tau(X) + C(X, PZ)\tau(Y) \\
&= C(X, \nabla^*_Y PZ) + X \cdot C(Y, PZ) - C(Y, \nabla^*_X PZ) - Y \cdot C(X, PZ) - C(\nabla_X Y - \nabla_Y X, PZ) \\
&\quad -C(Y, PZ)\tau(X) + C(X, PZ)\tau(Y) \\
&= X \cdot C(Y, PZ) - C(Y, \nabla^*_X PZ) - C(\nabla_X Y, PZ) - Y \cdot C(X, PZ) + C(X, \nabla^*_Y PZ) \\
&\quad +C(\nabla_Y X, PZ) - C(Y, PZ)\tau(X) + C(X, PZ)\tau(Y)
\end{aligned}
$$

$$
\overline{g}(\overline{R}_{XY}PZ, N) = (\nabla^*_X C)(Y, PZ) - (\nabla^*_Y C)(X, PZ) - C(Y, PZ)\tau(X) + C(X, PZ)\tau(Y)
$$

where

$$
\nabla^*_X C)(Y, PZ) = X \cdot C(Y, PZ) - C(\nabla_X Y, PZ) - C(Y, \nabla^*_X PZ).
$$

In Equation (3.1.22), by replacing Z by ξ we have

$$
\begin{aligned}
R_{XY}\xi &= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi \\
&= \nabla_X(-\tau(Y)\xi - \overset{\star}{A}_\xi(Y)) - \nabla_Y(-\tau(X)\xi - \overset{\star}{A}_\xi(X)) + \tau([X, Y])\xi + \overset{\star}{A}_\xi([X, Y]) \\
&= -X \cdot \tau(Y)\xi - \tau(Y)\nabla_X\xi - \nabla_X \overset{\star}{A}_\xi(Y) + Y \cdot \tau(X)\xi + \tau(X)\nabla_Y\xi + \nabla_Y \overset{\star}{A}_\xi(X) \\
&\quad +\tau([X, Y])\xi + \overset{\star}{A}_\xi([X, Y]) \\
&= -X \cdot \tau(Y)\xi - \tau(Y)(-\tau(X)\xi - \overset{\star}{A}_\xi(X)) - \nabla^*_X \overset{\star}{A}_\xi(Y) - C(X, \overset{\star}{A}_\xi(Y))\xi + Y \cdot \tau(X)\xi \\
&\quad +\tau(X)(-\tau(Y)\xi - \overset{\star}{A}_\xi(Y)) + \nabla^*_X \overset{\star}{A}_\xi(Y) + C(Y, \overset{\star}{A}_\xi(X))\xi + \tau([X, Y])\xi + \overset{\star}{A}_\xi([X, Y]) \\
&= (-X \cdot \tau(Y) - C(X, \overset{\star}{A}_\xi(Y)) + C(Y, \overset{\star}{A}_\xi(X)) + Y \cdot \tau(X) + \tau([X, Y]))\xi \\
&\quad +\tau(Y)\overset{\star}{A}_\xi(X) - \nabla^*_X \overset{\star}{A}_\xi(Y) - \tau(X)\overset{\star}{A}_\xi(Y) + \nabla^*_Y \overset{\star}{A}_\xi(X) + \overset{\star}{A}_\xi([X, Y])
\end{aligned}
$$

$$
\begin{aligned}
\overline{g}(\overline{R}_{XY}\xi, N) &= -X \cdot \tau(Y) - C(X, \overset{\star}{A}_\xi(Y)) + C(Y, \overset{\star}{A}_\xi(X)) + Y \cdot \tau(X) + \tau([X, Y]) \\
&= C(Y, \overset{\star}{A}_\xi(X)) - C(X, \overset{\star}{A}_\xi(Y)) - d\tau(X, Y).
\end{aligned}
$$

□

3.1.19 Definition. Let $x \in M$ and $PX \in S(TM)$ be a unitary vector field i.e $\overline{g}(PX, PX) = 1$, let $\sigma = Span(X, \xi)$ be a null plane contained in T_xM the null sectional curvature with respect to ξ of σ is given by:

$$
\begin{aligned}
K_\xi(\sigma) &= \overline{g}(\overline{R}_{\xi PX}PX, \xi) \\
&= (\nabla_\xi B)(PX, PX) - (\nabla_{PX} B)(\xi, PX) + \tau(\xi)B(PX, PX) - \tau(PX)B(\xi, PX) \\
&= (\nabla_\xi B)(PX, PX) - (\nabla_{PX} B)(\xi, PX) + \tau(\xi)B(PX, PX).
\end{aligned}
$$

3.1.20 Remark. If M is totally geodesic (i.e $B \equiv 0$) Then $K_\xi(\sigma) = 0$ and if M is totally umbilical (ie B$=\rho g$) then

$$
K_\xi(\sigma) = \xi(\rho) - \rho^2 + \tau(\xi)\rho.
$$

Proof.

$$
\begin{aligned}
K_\xi(\sigma) &= (\nabla_\xi B)(PX,PX) - (\nabla_{PX} B)(\xi,PX) + \tau(\xi)B(PX,PX) \\
&= \xi \cdot B(PX,PX) - 2B(\nabla_\xi PX,PX) - PX \cdot B(\xi,PX) + B(\nabla_{PX}\xi,PX) + B(\xi,\nabla_{PX}PX) \\
&\quad +\tau(\xi)B(PX,PX) \\
&= \xi(\rho) - 2\rho\overline{g}(\nabla_\xi PX,PX) + \rho\overline{g}(\nabla_{PX}\xi,PX) + \tau(\xi)\rho \\
\overline{g}(PX,PX) &= 1 \implies \xi \cdot \overline{g}(PX,PX) = 2\overline{g}(\overline{\nabla}_\xi PX,PX) = 2\overline{g}(\nabla_\xi PX,PX) = 0 \\
\overline{g}(\xi,PX) &= 0 \implies PX \cdot \overline{g}(\xi,PX) = 0 \implies \overline{g}(\overline{\nabla}_{PX}\xi,PX) = -\overline{g}(\overline{\nabla}_{PX}PX,\xi) = -B(PX,PX) = -\rho \\
K_\xi(\sigma) &= \xi(\rho) - \rho^2 + \tau(\xi)\rho.
\end{aligned}
$$

□

4. Necessary Condition for Coincidence between the Induced Connection ∇ and the Rigged Connection ∇_α

4.1 Condition for Coincidence between the Induced and the Rigged Connections for $\alpha = 1$

We know that ∇_1 is the unique torsion-free connection that is g_1-metric. Therefore $\nabla_1 = \nabla$ means that ∇ would also be g_1-metric.

- Let us find condition such that ∇ g_1-metric.

∇ is g_1-metric means that:

$$\nabla_X^{g_1}(Y,Z) \;=\; 0 \;\; \forall\; X,Y,Z \in \Gamma(TM). \tag{4.1.1}$$

From the decomposition (3.0.1), we have $\forall X \in TM, \quad X = PX + a\xi$ where $a = \overline{g}(X,N)$.

Therefore $X = PX + \eta(X)\xi$ where $\eta = \overline{g}(N,.)$ and $PX \in S(TM)$.

We have:

$$\nabla_X^{g_1}(Y,Z) \;=\; X.g_1(Y,Z) - g_1(\nabla_X^Y, Z) - g_1(Y,\nabla_X^Z). \tag{4.1.2}$$

We have:

$$\begin{aligned}
\nabla_X^{g_1}(Y,Z) &= X.\{g_1(PY+\eta(Y)\xi, PZ+\eta(Z)\xi)\} - g_1(\nabla_X(PY+\eta(Y)\xi), PZ+\eta(Z)\xi) \\
&\quad -g_1(PY+\eta(Y)\xi, \nabla_X(PZ+\eta(Z)\xi)) \\
&= X.\{g_1(PY,PZ) + g_1(PY,\eta(Z)\xi) + g_1(\eta(Y)\xi, PZ) + g_1(\eta(Y)\xi,\eta(Z)\xi\} \\
&\quad -g_1(\nabla_X(PY) + \nabla_X(\eta(Y)\xi)), PZ+\eta(Z)\xi) - g_1(PY+\eta(Y)\xi, \nabla_X(PZ) + \nabla_X(\eta(Z)\xi)) \\
&= X.\{\overline{g}(PY,PZ) + \eta(PY)\eta(Z) + \eta(Y)\eta(PZ) + \eta(Y)\eta(Z)\} - g_1(\nabla_X(PY), PZ) \\
&\quad -g_1(\nabla_X(\eta(Y)\xi), PZ) - g_1(\nabla_X PY, \eta(Z)\xi) - g_1(\nabla_X(\eta(Y)\xi), \eta(Z)\xi) \\
&\quad -g_1(\nabla_X(PZ), PY) - g_1(\nabla_X(\eta(Z)\xi), PY) - g_1(\nabla_X PZ, \eta(Y)\xi) - g_1(\nabla_X(\eta(Z)\xi), \eta(Y)\xi)
\end{aligned}$$

From Equation (3.1.2), $\eta(PX) = \overline{g}(N,PX) = 0 \; \forall X \in \Gamma(TM)$.

$$\begin{aligned}
\nabla_X^{g_1}(Y,Z) &= X.\{\overline{g}(PY,PZ) + \eta(Y)\eta(Z)\} - g_1(\nabla_X(PY), PZ) - g_1(\nabla_X(\eta(Y)\xi), PZ) \\
&\quad -g_1(\nabla_X PY, \eta(Z)\xi) - g_1(\nabla_X(\eta(Y)\xi), \eta(Z)\xi) - g_1(\nabla_X(PZ), PY) \\
&\quad -g_1(\nabla_X(\eta(Z)\xi), PY) - g_1(\nabla_X PZ, \eta(Y)\xi) - g_1(\nabla_X(\eta(Z)\xi), \eta(Y)\xi)
\end{aligned} \tag{4.1.3}$$

from Equation (3.1.17), $\forall X \;, Y \in \Gamma(TM) \quad \nabla_X PY = \nabla_X^* PY + C(X,PY)\xi$.

$$\begin{aligned}
g_1(\nabla_X PY, PZ) &= g_1(\nabla_X^* PY + C(X,PY)\xi, PZ) \\
&= g_1(\nabla_X^* PY, PZ) + C(X,PY) g_1(\xi, PZ) \\
&= g_1(\nabla_X^* PY, PZ) \\
&= \overline{g}(\nabla_X^* PY, PZ).
\end{aligned}$$

$$\begin{aligned}
g_1(\nabla_X(\eta(Y)\xi), PZ) &= g_1(X.\eta(Y)\xi + \eta(Y)\nabla_X\xi, PZ) \\
&= g_1(X.\eta(Y)\xi, PZ) + g_1(\eta(Y)\nabla_X\xi, PZ) \\
&= X.\eta(Y)g_1(\xi, PZ) + \eta(Y)g_1(\nabla_X\xi, PZ) \\
&= \eta(Y)g_1(-\tau(X)\xi - \overset{\star}{A}_\xi(X), PZ) \\
&= \eta(Y)g_1(-\overset{\star}{A}_\xi(X), PZ) \\
&= \eta(Y)\overline{g}(-\overset{\star}{A}_\xi(X), PZ).
\end{aligned}$$

$$\begin{aligned}
g_1(\nabla_X PY, \eta(Z)\xi) &= g_1(\nabla^*_X PY + C(X, PY)\xi, \eta(Z)\xi) \\
&= \eta(Z)g_1(\nabla^*_X PY, \xi) + \eta(Z)g_1(C(X, PY)\xi, \xi) \\
&= \eta(Z)C(X, PY).
\end{aligned}$$

$$\begin{aligned}
g_1(\nabla_X(\eta(Y)\xi), \eta(Z)\xi) &= g_1(X.\eta(Y)\xi + \eta(Y)\nabla_X\xi), \eta(Z)\xi) \\
&= X.\eta(Y)\eta(Z)g_1(\xi, \xi) + \eta(Y)\eta(Z)g_1(\nabla_X\xi, \xi) \\
&= X.\eta(Y)\eta(Z) + \eta(Y)\eta(Z)g_1(-\tau(X)\xi - \overset{\star}{A}_\xi(X), \xi) \\
&= X.\eta(Y)\eta(Z) - \eta(Y)\eta(Z)\tau(X).
\end{aligned}$$

By summing everything in Equation (4.1.3), we obtain:

$$\begin{aligned}
\nabla^{g_1}_X(Y, Z) &= X.\{\overline{g}(PY, PZ) + \eta(Y)\eta(Z)\} - \overline{g}(\nabla^*_X PY, PZ) - \eta(Y)\overline{g}(-\overset{\star}{A}_\xi(X), PZ) \\
&\quad -\eta(Z)C(X, PY) - X\cdot\eta(Y)\eta(Z) + \eta(Y)\eta(Z)\tau(X) - \overline{g}(\nabla^*_X PZ, PY) \\
&\quad -\eta(Z)\overline{g}(-\overset{\star}{A}_\xi(X), PY) - \eta(Y)C(X, PZ) - X\cdot\eta(Z)\eta(Y) + \eta(Y)\eta(Z)\tau(X) \\
&= X\cdot\overline{g}(PY, PZ) + X\cdot\eta(Y)\eta(Z) + X\cdot\eta(Z)\eta(Y) - \overline{g}(\nabla^*_X PY, PZ) - \eta(Y)\overline{g}(-\overset{\star}{A}_\xi(X), PZ) \\
&\quad -\eta(Z)C(X, PY) - X\cdot\eta(Y)\eta(Z) + \eta(Y)\eta(Z)\tau(X) - \overline{g}(\nabla^*_X PZ, PY) \\
&\quad -\eta(Z)\overline{g}(-\overset{\star}{A}_\xi(X), PY) - \eta(Y)C(X, PZ) - X\cdot\eta(Z)\eta(Y) + \eta(Y)\eta(Z)\tau(X) \\
&= X\cdot\overline{g}(PY, PZ) - \overline{g}(\nabla^*_X PY, PZ) + \eta(Y)\overline{g}(\overset{\star}{A}_\xi(X), PZ) - \eta(Z)C(X, PY) + \eta(Y)\eta(Z)\tau(X) \\
&\quad -\overline{g}(\nabla^*_X PZ, PY) + \eta(Z)\overline{g}(\overset{\star}{A}_\xi(X), PY) - \eta(Y)C(X, PZ) + \eta(Y)\eta(Z)\tau(X) \\
&= \{X.\overline{g}(PY, PZ) - \overline{g}(\nabla^*_X PY, PZ) - \overline{g}(\nabla^*_X PZ, PY)\} + \eta(Y)\overline{g}(\overset{\star}{A}_\xi(X), PZ) \\
&\quad -\eta(Z)C(X, PY) + \eta(Z)\overline{g}(\overset{\star}{A}_\xi(X), PY) - \eta(Y)C(X, PZ) + 2\eta(Y)\eta(Z)\tau(X)
\end{aligned}$$

We know that ∇^* is $\overline{g}$-metric, so we get:

$$\begin{aligned}
\nabla^{g_1}_X(Y, Z) &= \eta(Y)\overline{g}(\overset{\star}{A}_\xi(X), PZ) - \eta(Z)C(X, PY) + \eta(Z)\overline{g}(\overset{\star}{A}_\xi(X), PY) \\
&\quad -\eta(Y)C(X, PZ) + 2\eta(Y)\eta(Z)\tau(X) \\
&= \eta(Y)(\overline{g}(\overset{\star}{A}_\xi(X), PZ) - C(X, PZ)) + \eta(Z)(\overline{g}(\overset{\star}{A}_\xi(X), PY) - C(X, PY)) + 2\eta(Y)\eta(Z)\tau(X)
\end{aligned}$$

We have from Proposition 3.1.13 that $B(X, PY) = g(\overset{\star}{A}_\xi(X), PY)$, so finally we get

$$\begin{aligned}
\nabla^{g_1}_X(Y, Z) &= \eta(Y)(B(X, PZ) - C(X, PZ)) + \eta(Z)(B(X, PY) - C(X, PY)) \\
&\quad +2\eta(Y)\eta(Z)\tau(X). \qquad (4.1.4)
\end{aligned}$$

4.1.1 Theorem. *([2], page 3490)*

$$\nabla^{g_1}_X(Y,Z) = 0 \Leftrightarrow B(X,PZ) = C(X,PZ) \text{ and } \tau(X) = 0 \quad \forall X,Y,Z \in \Gamma(TM) \tag{4.1.5}$$

Proof. If $B(X,PZ) = C(X,PZ)$ and $\tau(X) = 0\ \forall\ X,Y,Z \in \Gamma(TM)$, from Equation (4.1.4), it is immediate that $\nabla^{g_1}_X(Y,Z) = 0$.

Now assume that we have $\nabla^{g_1}_X(Y,Z) = 0 \quad \forall X,Y,Z \in \Gamma(TM)$

• setting $Y = Z = \xi$ in (4.1.4), we have

$$\begin{aligned} 0 &= \nabla^{g_1}_X(\xi,\xi) = \eta(\xi)(B(X,0) - C(X,0)) + \eta(\xi)(B(X,0) - C(X,0)) \\ &\quad +2\eta(\xi)\eta(\xi)\tau(X) \\ 0 &= \tau(X). \end{aligned}$$

Then replacing Y to be ξ in (4.1.4), we have

$$\begin{aligned} 0 &= \nabla^{g_1}_X(\xi,Z) = \eta(\xi)(B(X,PZ) - C(X,PZ)) + \eta(Z)(B(X,0) - C(X,0)) \\ 0 &= \eta(\xi)(B(X,PZ) - C(X,PZ)) \\ &\quad B(X,PZ) = C(X,PZ). \end{aligned}$$

And we have proved the theorem. □

4.1.2 Remark. It may happen that the induced connection ∇ on M by $\overline{\nabla}$ is equal to the Levi-Civita connection of M for a given rigging, but if we make a change of rigging we loose this equality.

Let us consider another rigging ζ' for M and we decompose it as

$$\zeta' = \theta N + Z$$

where $Z \in \Gamma(TM)$ and $\theta \in C^\infty(M)$ is a function which never vanishes.

4.1.3 Proposition. The corresponding rigged ξ' and transverse vector field N' are given by:

$$\begin{aligned} \xi' &= \frac{1}{\theta}\xi \\ N' &= \theta N + V \quad \text{where } V = Z - \frac{1}{2\theta}\overline{g}(\zeta',\zeta')\xi. \end{aligned}$$

And we have the following relationships [1]:

$$\begin{aligned} B'(X,Y) &= \frac{1}{\theta}B(X,Y) && (4.1.6) \\ \tau'(X) &= \tau(X) + \frac{1}{\theta}\mathrm{d}\theta(X) + \frac{1}{\theta}B(V,X) && (4.1.7) \\ C'(X,PY) &= \theta C(X,PY) - \overline{g}(\overline{\nabla}_X V, PY) + \tau'(X)\overline{g}(V,PY) \\ \nabla'_X Y &= \nabla_X Y - \frac{1}{\theta}B(X,Y)V. \end{aligned}$$

4.1.4 Proposition. ([5], page 248)

Let ζ be a rigging for a totally geodesic null hypersurface such that $d\tau \neq 0$. Then, the induced and the rigged connections do not coincide for any election of the rigging.

Proof.

- $\nabla^N \neq \nabla$

If we assume that $\nabla^N = \nabla$, then from Theorem 4.1.1, it follows that $\tau = 0$. This implies that $\mathrm{d}\tau = 0$.

- Let ζ' be another selection of rigging. So we have from Equation (4.1.6) that $0 = B(X, PY) = \frac{1}{\theta}B'(X, PY)$. So Equation (4.1.7) implies that $\mathrm{d}\tau' = \mathrm{d}\tau + \mathrm{d}(\frac{1}{\theta})\mathrm{d}\theta = \mathrm{d}\tau \neq 0$.

Therefore we can conclude that the induced connection and the rigged connection (the Levi-Civita connection of M) do not coincide for any choice of rigging. □

Example of rigging for which there is coincidence.

4.1.5 Example. (**Monge Null Hypersurfaces of** $\mathbb{R}^3_1$, [4] page 11)

Let us consider $\mathbb{R}^3_1 = (\mathbb{R}^3, \overline{g})$ where $\overline{g} = -\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. Let D be an open subset of Riemannian manifold $\mathbb{R}^2_0 = (\mathbb{R}^2, \mathrm{d}u^2 + \mathrm{d}v^2)$. Let $f : D \to \mathbb{R}$ be a nowhere vanishing function and let us consider the immersion:

$$\begin{array}{rcl} i : D & \to & \mathbb{R}^3_1 \\ p = (u, v) & \mapsto & (x = f(p), y = u, z = v) \end{array}$$

$M = i(D)$ is called a Monge hypersurface of $\mathbb{R}^3_1$.

$$\forall p \in D,\ T_pM = Im\ i'(u, v) = \langle U = \begin{pmatrix} f'_u(p) \\ 1 \\ 0 \end{pmatrix}; V = \begin{pmatrix} f'_v(p) \\ 0 \\ 1 \end{pmatrix} \rangle.$$

Therefore $X = X^x\partial x + X^y\partial y + X^z\partial z \in T_pM$ if $X^x = X^u f'_u(p) + X^v f'_v(p)$. The vector field $\mathcal{N} = \frac{\partial}{\partial x} + f'_u(p)\frac{\partial}{\partial y} + f'_v(p)\frac{\partial}{\partial z}$ is normal to T_pM, indeed

$$\begin{aligned} \overline{g}\left(\frac{\partial}{\partial x} + f'_u(p)\frac{\partial}{\partial y} + f'_v(p)\frac{\partial}{\partial z}, f'_u(p)\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) &= -f'_u(p) + f'_u(p) = 0 \\ \overline{g}\left(\frac{\partial}{\partial x} + f'_u(p)\frac{\partial}{\partial y} + f'_v(p)\frac{\partial}{\partial z}, f'_v(p)\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right) &= -f'_v(p) + f'_v(p) = 0 \end{aligned}$$

4.1.6 Definition. M is called a Monge null hypersurface if $\mathcal{N}$ is a null vector field, ie

$$\begin{aligned} &\overline{g}(\mathcal{N}, \mathcal{N}) = 0 \\ &\overline{g}\left(\frac{\partial}{\partial x} + f'_u(p)\frac{\partial}{\partial y} + f'_v(p)\frac{\partial}{\partial z}, \frac{\partial}{\partial x} + f'_u(p)\frac{\partial}{\partial y} + f'_v(p)\frac{\partial}{\partial z}\right) = 0 \\ &(f'_u(p))^2 + (f'_v(p))^2 = 1. \end{aligned} \tag{4.1.8}$$

And in this case you can see that $\mathcal{N} = f'_u(p)U + f'_v(p)V \in T_pM$

If we take equation (4.1.8) and we derivate with respect to x^β $(\beta = 0, 1, 2)$, we obtain

$$f'_u(p)f''_{x^\alpha,u}(p) + f'_v(p)f''_{x^\alpha,v}(p) = 0. \tag{4.1.9}$$

Now we consider M a Monge null hypersurface and we endow M with the rigging $\mathcal{N}_f = \frac{1}{\sqrt{2}}\{-\frac{\partial}{\partial x} + f'_u\frac{\partial}{\partial y} + f'_v\frac{\partial}{\partial z}\}$. The vector field $\mathcal{N}_f$ is define on $\mathbb{R}^* \times D$ but is null only along M. The corresponding rigged vector field is $\xi_f = \frac{\mathcal{N}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\{\frac{\partial}{\partial x} + f'_u\frac{\partial}{\partial y} + f'_v\frac{\partial}{\partial z}\}$, the vector field ξ_f is define on $\mathbb{R}^* \times D$ but is null only along M. Indeed

$$\begin{aligned}\overline{g}(\mathcal{N}_f, \xi_f) &= \frac{1}{2}\overline{g}\left(-\frac{\partial}{\partial x} + f'_u\frac{\partial}{\partial y} + f'_v\frac{\partial}{\partial z}, \frac{\partial}{\partial x} + f'_u\frac{\partial}{\partial y} + f'_v\frac{\partial}{\partial z}\right) \\ &= \frac{1}{2}(1 + (f'_u)^2 + (f'_v)^2) = 1.\end{aligned}$$

$$x = f(u,v), \text{ so } \frac{\partial}{\partial u} = f'_u\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \text{ and } \frac{\partial}{\partial v} = f'_u\frac{\partial}{\partial x} + \frac{\partial}{\partial z} \implies TM = \langle\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\rangle$$

$\mathbb{R}^3_1$ is plate and we have

$$\begin{aligned}\overline{\nabla}_{\partial u}\xi &= \frac{1}{\sqrt{2}}\overline{\nabla}_{\partial u}\mathcal{N} \\ &= \frac{1}{\sqrt{2}}\overline{\nabla}_{f'_u\partial x + \partial y}\{\frac{\partial}{\partial x} + f'_u\frac{\partial}{\partial y} + f'_v\frac{\partial}{\partial z}\} \\ &= \frac{1}{\sqrt{2}}\overline{\nabla}_{\partial y}\{\frac{\partial}{\partial x} + f'_u\frac{\partial}{\partial y} + f'_v\frac{\partial}{\partial z}\} \\ &= \frac{1}{\sqrt{2}}(f''_{y,u}\frac{\partial}{\partial y} + f''_{y,v}\frac{\partial}{\partial z}) \\ &= \frac{1}{\sqrt{2}}(f''_{y,u}(\frac{\partial}{\partial u} - f'_u\frac{\partial}{\partial x}) + f''_{y,v}(\frac{\partial}{\partial v} - f'_v\frac{\partial}{\partial x})) \\ &= \frac{1}{\sqrt{2}}(f''_{y,u}\frac{\partial}{\partial u} + f''_{y,v}\frac{\partial}{\partial v}) + x(f'_u f''_{y,u} + f'_v f''_{y,v})\frac{\partial}{\partial x} \\ &= \frac{1}{\sqrt{2}}(f''_{y,u}\frac{\partial}{\partial u} + f''_{y,v}\frac{\partial}{\partial v}).\end{aligned}$$

We have

$$\begin{aligned}\overline{g}(\overline{\nabla}_{\partial u}\mathcal{N}, \mathcal{N}_f) &= \frac{1}{2}\overline{g}(f''_{y,u}\frac{\partial}{\partial y} + f''_{y,v}\frac{\partial}{\partial z}, -\frac{\partial}{\partial x} + f'_u\frac{\partial}{\partial y} + f'_v\frac{\partial}{\partial z}) \\ &= \frac{1}{2}\overline{g}(f'_u f''_{y,u} + f'_v f''_{y,v}) \\ &= 0 \text{ from Equation 4.1.9} \\ &\implies \overline{\nabla}_{\partial u}\mathcal{N} \in S(TM).\end{aligned}$$

Therefore since $\overline{\nabla}_{\partial u}\xi = -\tau(\frac{\partial}{\partial u})\xi - \overset{\star}{A}_\xi(\frac{\partial}{\partial u})$, we conclude that $\tau(\frac{\partial}{\partial u}) = 0$ and $\overset{\star}{A}_\xi(\frac{\partial}{\partial u}) = -\frac{1}{\sqrt{2}}(f''_{y,u}\frac{\partial}{\partial u} + f''_{y,v}\frac{\partial}{\partial v})$ and by the same computation, we will obtain: $\tau(\frac{\partial}{\partial v}) = 0$ and $\overset{\star}{A}_\xi(\frac{\partial}{\partial v}) = -\frac{1}{\sqrt{2}}(f''_{z,u}\frac{\partial}{\partial u} + f''_{z,v}\frac{\partial}{\partial v})$.

$$\tau(\frac{\partial}{\partial u}) = \tau(\frac{\partial}{\partial v}) = 0 \implies \tau = 0. \tag{4.1.10}$$

On the other hand, we have

$$\begin{aligned}
\mathcal{N}_f - \xi_f &= -\sqrt{2}\frac{\partial}{\partial x} \\
\Longrightarrow \overline{\nabla}_{\partial u}(\mathcal{N}_f - \xi_f) &= -\sqrt{2}\overline{\nabla}_{\partial u}(\frac{\partial}{\partial x}) \\
\Longrightarrow \overline{\nabla}_{\partial u}\mathcal{N}_f - \overline{\nabla}_{\partial u}\xi_f &= -\sqrt{2}\overline{\nabla}_{f'_u\partial x+\partial y}(\frac{\partial}{\partial x}) = 0 \\
\Longrightarrow \overline{\nabla}_{\partial u}\mathcal{N}_f = \overline{\nabla}_{\partial u}\xi_f &= -\overset{\star}{A}_\xi(\frac{\partial}{\partial u}) \\
\Longrightarrow A_{\mathcal{N}}(\frac{\partial}{\partial u}) &= \overset{\star}{A}_\xi(\frac{\partial}{\partial u}).
\end{aligned}$$

$$\begin{aligned}
\text{By the same computation, } A_{\mathcal{N}}(\frac{\partial}{\partial v}) &= \overset{\star}{A}_\xi(\frac{\partial}{\partial v}). \\
\text{Therefore: } \quad \overset{\star}{A}_\xi &= A_{\mathcal{N}} \quad \text{(conformal screen)} \\
\Longrightarrow \overline{g}(\overset{\star}{A}_\xi(X), Y) &= \overline{g}(A_{\mathcal{N}}(X), Y) \quad \forall X, Y \in \Gamma(TM) \\
\Longrightarrow C(X, PY) &= B(X, PY) \quad \forall X, Y \in \Gamma(TM) \qquad (4.1.11)
\end{aligned}$$

All the conditions of Theorem 4.1.1 are satisfied, so we conclude that the levi Civita connection of (M, g_1) and the induced connection on M coincide.

$$\nabla_1 = \nabla$$

Examples of Rigging for which this coincidence does not hold.

4.1.7 Example. The lightlike cone Λ_0^2 of $\mathbb{R}_1^3$ ([3], page 80)

Let consider the space $\mathbb{R}^3$ endowed with the metric $\overline{g} = -\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, $\mathbb{R}_1^3 = (\mathbb{R}^3, \overline{g})$ is called the Minkowski space of dimension 3.

$$\Lambda_0^2 = \{(x, y, z) \in \mathbb{R}^3; \ -x^2 + y^2 + z^2 = 0\}.$$

Let f be the submersion defined by,

$$\begin{aligned}
f : \mathbb{R}^3 &\to \mathbb{R} \\
(x, y, z) &\mapsto -x^2 + y^2 + z^2
\end{aligned}$$

$\Lambda_0^2 = f^{-1}(\{0\})$

Therefore, $\forall p = (x, y, z) \in \Lambda_0^2$, the tangent space at (x, y, z) is

$$T_p\Lambda_0^2 = \ker f'(x, y, z) = \{(a, b, c) \in \mathbb{R}^3, -a\sqrt{y^2 + z^2} + by + cz = 0\}$$

This tangent space is spanned by $Y = \frac{\partial}{\partial y} + \frac{y}{\sqrt{y^2+z^2}}\frac{\partial}{\partial x}$ and $Z = \frac{\partial}{\partial z} + \frac{z}{\sqrt{y^2+z^2}}\frac{\partial}{\partial x}$.

If we take $\xi = yY + zZ = \sqrt{y^2 + z^2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \in T_p\Lambda_0^2$ we have:

$$\begin{aligned}
\overline{g}(\xi, Y) &= \overline{g}\left(\sqrt{y^2 + z^2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + \frac{y}{\sqrt{y^2 + z^2}}\frac{\partial}{\partial x}\right) \\
&= -y + y = 0
\end{aligned}$$

$$\begin{aligned}
\text{And} \quad \overline{g}(\xi, Z) &= \overline{g}\left(\sqrt{y^2+z^2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \frac{\partial}{\partial z} + \frac{z}{\sqrt{y^2+z^2}}\frac{\partial}{\partial x}\right) \\
&= -z + z = 0 \\
&\implies \forall\, T \in T_p\Lambda_0^2,\ \overline{g}(\xi, T) = 0
\end{aligned}$$

Therefore we have proved that Λ_0^2 is a lightlike hypersurface of $\mathbb{R}_1^3$. Let us find the transversal vector field N such that $\overline{g}(N,N) = 0$ and $\overline{g}(\xi, N) = 1$.

$$\begin{aligned}
\text{Take} \quad N &= \frac{1}{2(y^2+z^2)}\left\{-\sqrt{y^2+z^2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right\} \quad \forall (x,y,z) \in \Lambda_0^2 \\
\text{and} \quad \overline{g}(N,N) &= \frac{1}{4(y^2+z^2)^2}(-y^2 - z^2 + y^2 + z^2) = 0 \\
\overline{g}(\xi, N) &= \frac{1}{2(y^2+z^2)}(y^2+z^2+y^2+z^2) = 1 \quad \text{and} \quad N \notin T\Lambda_0^2
\end{aligned}$$

It is obvious to see that the rigging is null only on the hypersurface and nowhere else, indeed

$$\begin{aligned}
\text{let} \quad p = (x,y,z) \notin \Lambda_0^2,\ N_p &= \frac{1}{2x^2}\{-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\} \\
\overline{g}_p(N_p, N_p) &= \frac{1}{4x^4}\overline{g}(-x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}) \\
&= \frac{1}{4x^4}(-x^2 + y^2 + z^2) \neq 0
\end{aligned}$$

The screen distribution $S(T\Lambda_0^2)$ of $T\Lambda_0^2$ is of dimension 1, and it is such that

$$\forall\, T \in S(T\Lambda_0^2),\ \overline{g}(T, N) = 0$$

$$\begin{aligned}
\text{Therefore we can take} \quad S(T\Lambda_0^2) &= \langle S \rangle \qquad \text{where } S = zY - yZ \ \in T\Lambda_0^2 \\
\text{And we see that} \quad \overline{g}(S, N) &= \frac{1}{2(y^2+z^2)}\overline{g}\left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, -\sqrt{y^2+z^2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) \\
&= \frac{1}{2(y^2+z^2)}(yz - zy) = 0
\end{aligned}$$

it is obvious to see that all the Christoffel of the metric vanishes. Therefore,

$$\begin{aligned}
\forall T = gY + hZ \in T\Lambda_0^2,\ \nabla_T\xi &= \nabla_T(\sqrt{y^2+z^2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}) \\
&= T(x)\frac{\partial}{\partial x} + T(y)\frac{\partial}{\partial y} + T(z)\frac{\partial}{\partial z} \\
&= (g\frac{y}{\sqrt{y^2+z^2}} + h\frac{y}{\sqrt{y^2+z^2}})\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z} \\
&= gX + hY = T \\
\forall T \in T\Lambda_0^2,\ \nabla_T\xi &= T \implies \nabla_\xi\xi = \xi \notin S(T\Lambda_0^2) \implies \tau(\xi) \neq 0
\end{aligned}$$

We can conclude that there is no coincidence between the rigged connection and the induced connection of the metric.

4.1.8 Example. Twisted Product ([5], page 249)

Consider $(\mathbb{R}^2, 2\mathrm{d}u\mathrm{d}v)$ and take (Q, g_Q) any Riemannian manifold. Consider the twisted product $(\overline{M}, \overline{g}) = (Q \times \mathbb{R}^2, g_Q + 2f^2(x,u,v)\mathrm{d}u\mathrm{d}v)$, being $f \in C^\infty(Q \times \mathbb{R}^2)$ a positive function with $w(\frac{f_v}{f}) \neq 0$ for some $w \in T_{x_0}Q$.

Let $M = \{(x,u,v) \in Q \times \mathbb{R}^2 : u = u_0\}$ and we consider the submersion

$$\begin{array}{rcl} f : Q \times \mathbb{R}^2 & \to & \mathbb{R} \\ (x,u,v) & \mapsto & u - u_0 \end{array}$$

$M = f^{-1}(\{0\})$, hence $\forall p = (x,u,v) \in M,\ \mathrm{d}f(p) = \mathrm{d}u$, let $X = X_x + X^u\partial u + X^v\partial v \in T_pQ \times \mathbb{R}^2$, $X \in T_pM$ iff $\mathrm{d}u(X) = 0 \implies X^u = 0$

$$X \in T_pM \quad \text{iff} \quad X = X_x + X^v\partial v.$$

The vector field $N = \frac{1}{2f^2}\partial u$ is a null rigging for M, indeed we have $\mathrm{d}u(N) = \frac{1}{2f^2} \neq 0$, it follows that N is not a section of TM and $\overline{g}(N,N) = g_Q(N,N) = 0$.

The corresponding rigged vector field is $\xi = \partial v$ and the screen distribution is $S \approx TQ$.

We can compute the Christoffel of $\overline{g}$ since $\overline{g}$ is a semi-Riemannian metric.

the metric $\overline{g}$ and its inverse $\overline{g}^{-1}$ are given by:

$$\overline{g} = \begin{pmatrix} g_Q & 0 & 0 \\ 0 & 0 & 2f^2 \\ 0 & 2f^2 & 0 \end{pmatrix} \qquad \overline{g}^{-1} = \begin{pmatrix} g_Q^{-1} & 0 & 0 \\ 0 & 0 & \frac{1}{2f^2} \\ 0 & \frac{1}{2f^2} & 0 \end{pmatrix}$$

We know that

$$\begin{aligned} \overline{\nabla}_\xi \xi = \nabla_\xi \xi &= -\tau(\xi)\xi - \overset{\star}{A}_\xi(\xi) \\ &= -\tau(\xi)\xi \text{ since } \overset{\star}{A}_\xi(\xi) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \overline{\nabla}_\xi \xi &= \overline{\nabla}_{\partial v}\partial v = \Gamma^k_{vv}\partial k = \Gamma^v_{vv}\partial v \\ \Gamma^v_{vv} &= \frac{1}{2}g^{vu}\{\partial v g_{uv} + \partial v g_{vu} - \partial u g_{vv}\} \\ &= \frac{1}{4f^2}\{2\partial v g_{uv}\} \\ &= \frac{2f_v}{f}. \end{aligned}$$

So we conclude that $\tau(\xi) = -\frac{2f_v}{f}$.

Let $PX \in TQ$, let $\{x_i\}$ be a local coordinate system of TQ and $\{\partial i\}$ be a basis of TQ.

We know that $\nabla_{PX}\xi = -\tau(PX)\xi - \overset{\star}{A}_\xi(PX)$.

On the other hand,

$$\begin{aligned}
PX &= PX^i \partial x_i \\
\overline{\nabla}_{PX}\xi &= \nabla_{PX}\xi = PX^i \nabla_{\partial x_i}\partial v \\
&= PX^i \Gamma^k_{iv}\partial k \\
&= PX^i(\Gamma^v_{iv}\partial v + \Gamma^i_{iv}\partial i)
\end{aligned}$$

$$\begin{aligned}
\Gamma^v_{iv} &= \frac{1}{2}g^{vu}\{\partial i g_{uv} + \partial v g_{iu} - \partial u g_{iv}\} \\
&= \frac{1}{4f^2}\{\partial i(2f^2)\} \\
&= \frac{\partial i(f)}{f} \\
\Gamma^i_{iv} &= \frac{1}{2}g^{ij}\{\partial i g_{jv} + \partial v g_{ij} - \partial j g_{iv}\} \\
&= 0
\end{aligned}$$

$$\text{Therefore , } \nabla_{PX}\xi = PX^i \frac{\partial i(f)}{f}\xi = \frac{PX(f)}{f}\xi.$$

So, we conclude that $\overset{\star}{A}_\xi(PX) = 0$ and $\tau(PX) = -\frac{PX(f)}{f} \quad \forall X \in \Gamma(TM)$.

$$\begin{aligned}
B(PX, Y) &= \overline{g}(\overset{\star}{A}_\xi(PX), Y) = 0 \text{ and } B(\xi, Y) = 0 \;\forall X, Y \in \Gamma(TM) \\
B(X, Y) &= 0 \;\; \forall\; X, Y \in \Gamma(TM) \;\; \text{therefore } M \text{ is totally geodesic.}
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\mathrm{d}\tau(\xi, w) &= \xi \cdot \tau(w) - w \cdot \tau(\xi) - \tau([\xi, w]) \\
&= \xi \cdot \tau(w) - w \cdot \tau(\xi) \\
&= -\xi \cdot \left(\frac{w(f)}{f}\right) + 2w \cdot (\frac{f_v}{f}) \\
&= -w \cdot \left(\frac{\xi(f)}{f}\right) + 2w \cdot (\frac{f_v}{f}) = w \cdot (\frac{f_v}{f}) \neq 0 \;\; \text{because } [\xi, w] = 0.
\end{aligned}$$

We have M totally geodesic and $\mathrm{d}\tau \neq 0$, using Proposition 4.1.4, we conclude that the induced connection and the rigged connection do not coincide for any choice of the rigging.

4.2 Coincidence between the Induced and the Rigged Connections for $\alpha \neq 1$

In this chapter, we are interested in the general case where $\alpha \neq 0$ is a function and N is a closed rigging and we want to find conditions such that $\nabla_\alpha = \nabla$, where ∇ is the connection on M obtained through the projection of $\overline{\nabla}$ along N and ∇_α is the Levi-Civita connection of M with respect to the metric g_α.

4.2.1 Definition. The screen distribution $S(TM)$ is said to be totally umbilical if there exists $f \in C^\infty(M)$ such that $C(X, PY) = fg(X, Y) \ \ \forall X, Y \in TM$, If $C = 0$ then $S(TM)$ is said to be totally geodesic.

We say that the rigging N is conformal if there exists $\phi \in C^\infty(M), \phi(x) \neq 0 \ \forall x \in M$ such that $A_N = \phi \overset{\star}{A}_\xi$. When $d\overline{\eta} = 0$ we say that the rigging N is closed or that (M, g, N) is a closed rigged null hypersurface.

4.2.2 Definition (α-associated metric)**.** We define the α-associated metric of (M, g, N) by :

$$g_\alpha = g + \alpha\eta \otimes \eta. \tag{4.2.1}$$

And we recall from Definition 3.1.4 that The vector field ξ over M is g_1-metrically equivalent to η i.e ($\eta = g_1(\xi, .)$.

4.2.3 Proposition. The α-associated metric g_α is non-degenerate.

Proof. Let $X \in \Gamma(TM)$ such that $g_\alpha(X, Y) = 0 \ \ \forall \ Y \in \Gamma(TM)$;

In particular for $\xi \in Rad(TM) \subset TM$,$0 = g_\alpha(X, \xi) = g(X, \xi) + \alpha\eta(X)\eta(\xi) = \alpha\eta(X)\eta(\xi)$, it follows that $\eta(X) = 0$, so $X \in S(TM)$. Therefore, we will have $g_\alpha(X, Y) = g(X, Y) = 0 \ \ \forall \ Y \in \Gamma(TM)$, thus $X \in Rad(TM) \cap S(TM)$. So $X = 0$, therefore g_α is non degenerate. □

We recall the Koszul Identity:

The Levi-Civita connection ∇_α with respect to the metric g_α of M satisfies:

$$\begin{aligned} 2g_\alpha(\nabla_{\alpha U} V, W) &= U \cdot g_\alpha(V, W) + V \cdot g_\alpha(U, W) - W \cdot g_\alpha(U, V) + g_\alpha([U, V], W) \\ &\quad + g_\alpha([W, U], V) - g_\alpha([V, W], U) \end{aligned} \tag{4.2.2}$$

4.2.4 Proposition. When N is closed, we have

$$\overline{g}(\overline{\nabla}_U N, V) = \overline{g}(U, \overline{\nabla}_V N) \ \ \forall U, V \in \Gamma(T\overline{M}). \tag{4.2.3}$$

Proof.

$$\begin{aligned} 0 = d\overline{\eta}(U, V) &= U \cdot \overline{\eta}(V) - V \cdot \overline{\eta}(U) - \overline{\eta}([U, V]) \\ &= U \cdot \overline{g}(N, V) - V \cdot \overline{g}(N, U) - \overline{\eta}(\overline{\nabla}_U V - \overline{\nabla}_V U) \\ &= \overline{g}(\overline{\nabla}_U N, V) + \overline{g}(N, \overline{\nabla}_U V) - \overline{g}(\overline{\nabla}_V N, U) - \overline{g}(N, \overline{\nabla}_V U) - \overline{g}(N, \overline{\nabla}_U V - \overline{\nabla}_V U) \\ &= \overline{g}(\overline{\nabla}_U N, V) - \overline{g}(\overline{\nabla}_V N, U) \end{aligned}$$

□

4.2.5 Lemma. [4, page 4] If (M, g, N) is a closed rigged null hypersurface with a conformal screen distribution, then the 1-form τ vanishes on the screen distribution.

Proof. Using Proposition 4.2.4, and replacing $U = \xi$ and $V = PX$, we have

$$\begin{aligned} \overline{g}(\overline{\nabla}_\xi N, PX) &= \overline{g}(\xi, \overline{\nabla}_{PX} N) \\ \overline{g}(\tau(\xi)N - A_N(\xi), PX) &= \overline{g}(\xi, \tau(PX)N - A_N(PX)) \\ -\overline{g}(A_N(\xi), PX) &= \tau(PX) \\ -\phi\overline{g}(\overset{\star}{A}_\xi(\xi), PX) &= \tau(PX) \end{aligned}$$

$$
\begin{aligned}
-\phi B(\xi, PX) &= \tau(PX) \\
\tau(PX) = 0.
\end{aligned}
$$

□

4.2.6 Proposition. [4, page 6](Relation between ∇_α and ∇)

Let ∇_α be the Levi-Civita connection on TM with respect to the metric g_α and ∇ the induced connection on M from $\overline{\nabla}$. Then both are linked by:

$$
\begin{aligned}
\nabla_{\alpha X} Y &= \nabla_X Y - \frac{1}{2}\eta(X)\eta(Y)(d\alpha)^{\#g_\alpha} + \frac{\alpha}{2}[\eta(X)(i_Y d\eta)^{\#g_\alpha} + \eta(Y)(i_X d\eta)^{\#g_\alpha}] \\
&+ \frac{1}{2\alpha}[\alpha L_N \overline{g}(X,Y) + 2B(X,Y) + d\alpha(X)\eta(Y) + d\alpha(Y)\eta(X)]\xi. \qquad (4.2.4)
\end{aligned}
$$

Proof. Using Equation (4.2.2) and Equation (4.2.1),

$$
\begin{aligned}
2g_\alpha(\nabla_{\alpha X} Y, Z) &= X \cdot g_\alpha(Y,Z) + Y \cdot g_\alpha(X,Z) - Z \cdot g_\alpha(X,Y) + g_\alpha([X,Y],Z) \\
&+ g_\alpha([Z,X],Y) - g_\alpha([Y,Z],X) \\
&= X \cdot [g(Y,Z) + \alpha\eta(Y)\eta(Z)] + Y \cdot [g(X,Z) + \alpha\eta(X)\eta Z)] - Z \cdot [g(X,Y) + \alpha\eta(X)\eta(Y)] \\
&+ g([X,Y],Z) + \alpha\eta([X,Y])\eta(Z) + g([Z,X],Y) + \alpha\eta([Z,X]\eta(Y) - g([Y,Z],X) \\
&- \alpha\eta([Y,Z])\eta(X) \\
&= 2\overline{g}(\overline{\nabla}_X Y, Z) + d\alpha(X)\eta(Y)\eta(Z) + \alpha X \cdot (\eta(Y))\eta(Z) + \alpha X \cdot (\eta(Z)))\eta(Y) + d\alpha(Y)\eta(X)\eta(Z) \\
&+ \alpha Y \cdot (\eta(X))\eta(Z) + \alpha Y \cdot (\eta(Z))\eta(X) - d\alpha(Z)\eta(X)\eta(Y) - \alpha Z \cdot (\eta(X))\eta(Y) \\
&- \alpha Z \cdot (\eta(Y))\eta(X) + \alpha\eta([X,Y])\eta(Z) + \alpha\eta([Z,X])\eta(Y) - \alpha\eta([Y,Z])\eta(X) \\
&= 2\overline{g}(\nabla_X Y, Z) + 2B(X,Y)\eta(Z) + d\alpha(X)\eta(Y)\eta(Z) + d\alpha(Y)\eta(X)\eta(Z) + \alpha d\eta(Y,Z)\eta(X) \\
&+ \alpha d\eta(X,Z)\eta(Y) + \alpha[X \cdot (\eta(Y)) + Y \cdot (\eta(X))]\eta(Z) - d\alpha(Z)\eta(X)\eta(Y) \\
&+ \alpha\eta(\nabla_X Y - \nabla_Y X)\eta(Z) \\
&= 2g_\alpha(\nabla_X Y, Z) - 2\alpha\eta(\nabla_X Y)\eta(Z) + 2B(X,Y)\eta(Z) + d\alpha(X)\eta(Y)\eta(Z) + d\alpha(Y)\eta(X)\eta(Z) \\
&+ \alpha d\eta(Y,Z)\eta(X) + \alpha d\eta(X,Z)\eta(Y) + \alpha L_N\overline{g}(X,Y)\eta(Z) - d\alpha(Z)\eta(X)\eta(Y) \\
&+ 2\alpha\eta(\nabla_X Y)\eta(Z) \\
&= 2g_\alpha(\nabla_X Y, Z) + 2B(X,Y)\eta(Z) + d\alpha(X)\eta(Y)\eta(Z) + d\alpha(Y)\eta(X)\eta(Z) \\
&+ \alpha d\eta(Y,Z)\eta(X) + \alpha d\eta(X,Z)\eta(Y) + \alpha L_N\overline{g}(X,Y)\eta(Z) - d\alpha(Z)\eta(X)\eta(Y) \\
&\text{Now we use the fact that } g_\alpha(X,\xi) = \alpha\eta(X) \\
&= 2g_\alpha(\nabla_X Y, Z) + \frac{2}{\alpha}B(X,Y)g_\alpha(\xi,Z) + \frac{1}{\alpha}g_\alpha(d\alpha(X)\eta(Y)\xi + d\alpha(Y)\eta(X)\xi + \alpha L_N\overline{g}(X,Y)\xi, Z) \\
&+ \alpha\eta(X)g_\alpha((i_Y d\eta)^{\#g_\alpha}, Z) + \alpha\eta(Y)g_\alpha((i_X d\eta)^{\#g_\alpha}, Z) - \eta(X)\eta(Y)g_\alpha(d\alpha^{\#g_\alpha}, Z).
\end{aligned}
$$

And we have the result. □

4.2.7 Proposition. Since N is a closed rigging, then

$$
L_N\overline{g}(X,Y) = 2\tau(X)\eta(Y) - 2C(X,PY). \qquad (4.2.5)
$$

Proof.

$$\begin{aligned}
L_N\overline{g}(X,Y) &= N\cdot\overline{g}(X,Y)-\overline{g}([N,X],Y)-\overline{g}(X,[N,Y]) \\
&= \overline{g}(\overline{\nabla}_N X,Y)+\overline{g}(X,\overline{\nabla}_N Y)-\overline{g}(\overline{\nabla}_N X-\overline{\nabla}_X N,Y)-\overline{g}(X,\overline{\nabla}_N Y-\overline{\nabla}_Y N) \\
&= \overline{g}(\overline{\nabla}_X N,Y)+\overline{g}(X,\overline{\nabla}_Y N)=2\overline{g}(\overline{\nabla}_X N,Y) \quad \text{from Proposition 4.2.4} \\
&= 2\overline{g}(\tau(X)N-A_N(X),Y) \\
&= 2\tau(X)\eta(Y)-2C(X,PY) \quad \text{from Proposition 3.1.17}
\end{aligned}$$

□

Therefore, Equation (4.2.4) becomes

$$\begin{aligned}
\nabla_{\alpha X}Y &= \nabla_X Y-\frac{1}{2}\eta(X)\eta(Y)(d\alpha)^{\#g_\alpha} \\
&+\frac{1}{2\alpha}[2\alpha\tau(X)\eta(Y)-2\alpha C(X,PY)+2B(X,Y)+d\alpha(X)\eta(Y)+d\alpha(Y)\eta(X)]\xi
\end{aligned} \tag{4.2.6}$$

4.2.8 Theorem. *[4, page 7] Let (M,g,N) be a closed rigged null hypersurface,*

1. *Let α such that $d\alpha(PX)=0 \quad \forall X\in\Gamma(TM)$ i.e α is constant on each leaf of the screen distribution, then the induced connection $\nabla=\nabla_\alpha$ if and only if:*

$$\overset{\star}{A}_\xi=\alpha A_N \quad \text{and} \quad 2\alpha\tau(\xi)+d\alpha(\xi)=0. \tag{4.2.7}$$

2. *Let α be a non null real number, then the induced connection $\nabla=\nabla_\alpha$ if and only if*

$$\overset{\star}{A}_\xi=\alpha A_N \quad \text{and} \quad \tau\equiv 0. \tag{4.2.8}$$

Proof. 1.

$$\begin{aligned}
X &= PX+\eta(X)\xi \\
d\alpha(X) &= \eta(X)d\alpha(\xi) \implies \alpha d\alpha=d\alpha(\xi)g_\alpha(\xi,\cdot) \\
\alpha d\alpha^{\#g_\alpha} &= d\alpha(\xi)\xi.
\end{aligned}$$

Therefore Equation (4.2.6) becomes

$$\begin{aligned}
\nabla_{\alpha X}Y &= \nabla_X Y-\frac{1}{2\alpha}\eta(X)\eta(Y)d\alpha(\xi)\xi \\
&+\frac{1}{2\alpha}[2\alpha\tau(X)\eta(Y)-2\alpha C(X,PY)+2B(X,Y)+\eta(X)\eta(Y)d\alpha(\xi)+\eta(Y)\eta(X)d\alpha(\xi)]\xi \\
&= \nabla_X Y+\frac{1}{2\alpha}[2\alpha\tau(X)\eta(Y)-2\alpha C(X,PY)+2B(X,Y)+\eta(X)\eta(Y)d\alpha(\xi)]\xi
\end{aligned}$$

$\nabla=\nabla_\alpha$ if and only if

$$2\alpha\tau(X)\eta(Y)-2\alpha C(X,PY)+2B(X,Y)+\eta(X)\eta(Y)d\alpha(\xi)=0 \quad \forall X,Y\in\Gamma(TM) \tag{4.2.9}$$

especially for $X=Y=\xi$ we obtain

$$2\alpha\tau(\xi)+d\alpha(\xi)=0$$

now for $Y = \xi$ we obtain

$$2\alpha\tau(X) + \eta(X)d\alpha(\xi) = 0 \quad \forall X \in \Gamma(TM).$$

Back to Equation (4.2.9), we have

$$\begin{aligned}
-2\alpha C(X, PY) + 2B(X, Y) = 0 &\ \ \forall X, Y \in \Gamma(TM)\\
g(\overset{\star}{A}_\xi(X), Y) = \alpha g(A_N(X), Y) &\ \ \forall X, Y \in \Gamma(TM)\\
\overline{g}(\overset{\star}{A}_\xi(X) - \alpha A_N(X), Y) &\ \ \forall X, Y \in \Gamma(TM)\\
\text{but we also have that} \quad \overline{g}(\overset{\star}{A}_\xi(X) - \alpha A_N(X), N) &= 0\\
\overline{g}(\overset{\star}{A}_\xi(X) - \alpha A_N(X), U) = 0 \ \ \forall X \in \Gamma(TM), &\ \ \forall U \in \Gamma(T\overline{M})\\
\Longrightarrow \ \overset{\star}{A}_\xi(X) = \alpha A_N(X) &\ \ \forall X \in \Gamma(TM).
\end{aligned}$$

Now suppose that we have $\overset{\star}{A}_\xi = \alpha A_N$ and $2\alpha\tau(\xi) + d\alpha(\xi) = 0$, using Lemma 4.2.5 we conclude that $\tau(PX) = 0 \ \forall \ X \in \Gamma(TM)$, therefore

$$\begin{aligned}
&2\alpha\tau(X)\eta(Y) - 2\alpha C(X, PY) + 2B(X, Y) + \eta(X)\eta(Y)d\alpha(\xi)\\
&= 2\alpha\tau(X)\eta(Y) + \eta(X)\eta(Y)d\alpha(\xi)\\
&= \eta(Y)(2\alpha\tau(X) + \eta(X)d\alpha(\xi))\\
&= \eta(Y)(2\alpha\tau(PX) + 2\alpha\eta(X)\tau(\xi) + \eta(X)d\alpha(\xi))\\
&= \eta(Y)\eta(X)(2\alpha\tau(\xi) + d\alpha(\xi))\\
&= 0.
\end{aligned}$$

So Equation (4.2.9) holds.

2. α is a non zero real number, using part 1 of the theorem, we have

$$\begin{aligned}
\nabla_\alpha = \nabla \quad &\text{iff} \quad \overset{\star}{A}_\xi = \alpha A_N \text{ and } 2\alpha\tau(\xi) + d\alpha(\xi) = 0\\
&\text{iff} \quad \overset{\star}{A}_\xi = \alpha A_N \text{ and } \tau(\xi) = 0\\
\text{using Lemma 4.2.5} \quad & \quad \tau(PX) = 0 \ \forall \ X \in \Gamma(TM)\\
\text{therefore, } \nabla_\alpha = \nabla \quad &\text{iff} \quad \overset{\star}{A}_\xi = \alpha A_N \text{ and } \tau(X) = 0 \ \forall \ X \in \Gamma(TM).
\end{aligned}$$

.

□

4.2.9 Proposition. Let (M,g,N) be a rigged null hypersurface. If α is constant on each leaf of the screen distribution such that it satisfies Theorem 4.2.8, then Theorem 4.2.8 holds for any change of rigging $\tilde{N} = \psi N$, with ψ a non vanishing function and for $\tilde{\alpha} = \frac{\alpha}{\psi^2}$.

Proof. Let us first remark that

$$\begin{aligned}
\widetilde{\nabla}_X Y = \nabla_X Y \quad &\forall \ X, Y \in \Gamma(TM)\\
\text{and} \ \ g_{\tilde{\alpha}} = g_\alpha &
\end{aligned}$$

$\nabla_{\tilde{\alpha}}$ is the Levi-Civita connection of $g_{\tilde{\alpha}}$ and ∇_α is the Levi-Civita connection of g_α, so $\nabla_{\tilde{\alpha}} = \nabla_\alpha = \nabla^N$. Therefore $\nabla_{\tilde{\alpha}} = \nabla^{\tilde{N}}$ if and only if and only if $\nabla^{\tilde{N}} = \nabla^N$ which is already the case. So we have proved that

$$\nabla_{\tilde{\alpha}} = \nabla^{\tilde{N}}.$$

□

This proposition means that if the couple (N, α) ensures the coincidence between the rigged and the induced connections then for any non vanishing function ψ, the couple $(\psi N, \frac{\alpha}{\psi^2})$ also ensures the coincidence. And by induction all the couples $(\psi^p N, \frac{\alpha}{\psi^{2p}}), \quad p \in \mathbb{N}$.

In the particular case of $\alpha = 1$, if the rigged connection of g_1 coincides with the induced connection, so does the rigged connection of any $g_t, t \in \mathbb{R}_+^*$.

4.2.10 Remark. For some situations, we can have a couple $(\tilde{N}, \tilde{\alpha})$ where $\tilde{\alpha}$ is a function which is not constant along the screen distribution and $\tilde{N}$ a non-closed rigging and we still have the coincidence between the induced and the rigged connections.

4.2.11 Example. (**Monge Null Hypersurfaces of** $\mathbb{R}_1^3$, [4] page 14)

Let us consider M the Monge null hypersurface of Example 4.1.5 and we endow M with the rigging $\mathcal{N}_f = \frac{1}{2x}\{\frac{\partial}{\partial x} - f'_u\frac{\partial}{\partial y} - f'_v\frac{\partial}{\partial z}\}$. The vector field $\mathcal{N}_f$ is defined on $\mathbb{R}^* \times D$ but is null only along M. The corresponding rigged vector field is $\xi_f = -x\mathcal{N} = x\{-\frac{\partial}{\partial x} - f'_u\frac{\partial}{\partial y} - f'_v\frac{\partial}{\partial z}\}$, the vector field ξ_f is define on $\mathbb{R}^* \times D$ but is null only along M. Indeed

$$\begin{aligned}
\overline{g}(\mathcal{N}_f, \xi_f) &= \frac{1}{2}\overline{g}\left(\frac{\partial}{\partial x} - f'_u\frac{\partial}{\partial y} - f'_v\frac{\partial}{\partial z}, -\frac{\partial}{\partial x} - f'_u\frac{\partial}{\partial y} - f'_v\frac{\partial}{\partial z}\right)\\
&= \frac{1}{2}(1 + (f'_u)^2 + (f'_v)^2) = 1.
\end{aligned}$$

$$x = f(u,v), \text{ so } \frac{\partial}{\partial u} = f'_u\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \text{ and } \frac{\partial}{\partial v} = f'_u\frac{\partial}{\partial x} + \frac{\partial}{\partial z} \implies TM = \langle\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\rangle$$

$\mathbb{R}_1^3$ is plate and we have:

$$\begin{aligned}
\overline{\nabla}_{\partial u}\xi &= -\overline{\nabla}_{\partial u}x\mathcal{N}\\
&= -\partial u(x)\mathcal{N} - x\overline{\nabla}_{\partial u}\mathcal{N}\\
&= \frac{f'_u}{x}\xi - x\overline{\nabla}_{f'_u\partial x + \partial y}\{\frac{\partial}{\partial x} + f'_u\frac{\partial}{\partial y} + f'_v\frac{\partial}{\partial z}\}\\
&= \frac{f'_u}{x}\xi - x\overline{\nabla}_{\partial y}\{\frac{\partial}{\partial x} + f'_u\frac{\partial}{\partial y} + f'_v\frac{\partial}{\partial z}\}\\
&= \frac{f'_u}{x}\xi - x(f''_{y,u}\frac{\partial}{\partial y} + f''_{y,v}\frac{\partial}{\partial z})\\
&= \frac{f'_u}{x}\xi - x(f''_{y,u}(\frac{\partial}{\partial u} - f'_u\frac{\partial}{\partial x}) + f''_{y,v}(\frac{\partial}{\partial v} - f'_v\frac{\partial}{\partial x}))\\
&= \frac{f'_u}{x}\xi - x(f''_{y,u}\frac{\partial}{\partial u} + f''_{y,v}\frac{\partial}{\partial v}) + x(f'_u f''_{y,u} + f'_v f''_{y,v})\frac{\partial}{\partial x}\\
&= \frac{f'_u}{x}\xi - x(f''_{y,u}\frac{\partial}{\partial u} + f''_{y,v}\frac{\partial}{\partial v}).
\end{aligned}$$

We have

$$\begin{aligned}
\overline{g}(\overline{\nabla}_{\partial u}\mathcal{N},\mathcal{N}_f) &= -\frac{1}{2}\overline{g}(f''_{y,u}\frac{\partial}{\partial y}+f''_{y,v}\frac{\partial}{\partial z},\frac{\partial}{\partial x}-f'_u\frac{\partial}{\partial y}-f'_v\frac{\partial}{\partial z}) \\
&= \frac{1}{2}\overline{g}(f'_uf''_{y,u}+f'_vf''_{y,v}) \\
&= 0 \text{ from Equation (4.1.9)} \\
&\implies \overline{\nabla}_{\partial u}\mathcal{N} \in S(TM).
\end{aligned}$$

Therefore since $\overline{\nabla}_{\partial u}\xi = -\tau(\frac{\partial}{\partial u})\xi - \overset{\star}{A}_\xi(\frac{\partial}{\partial u})$, we conclude that $\tau(\frac{\partial}{\partial u}) = -\frac{f'_u}{x}$ and $\overset{\star}{A}_\xi(\frac{\partial}{\partial u}) = x(f''_{y,u}\frac{\partial}{\partial u} + f''_{y,v}\frac{\partial}{\partial v})$ and by the same computation, we will obtain: $\tau(\frac{\partial}{\partial v}) = -\frac{f'_v}{x}$ and $\overset{\star}{A}_\xi(\frac{\partial}{\partial v}) = x(f''_{z,u}\frac{\partial}{\partial u} + f''_{z,v}\frac{\partial}{\partial v})$.
On the other hand:

$$\begin{aligned}
\overline{\nabla}_{\partial u}\mathcal{N}_f &= \overline{\nabla}_{\partial u}\frac{1}{2x}\{\frac{\partial}{\partial x}-f'_u\frac{\partial}{\partial y}-f'_v\frac{\partial}{\partial z}\} \\
&= \partial u(\frac{1}{2x})(2x\mathcal{N}_f)+\frac{1}{2x}\overline{\nabla}_{\partial u}\{\frac{\partial}{\partial x}-f'_u\frac{\partial}{\partial y}-f'_v\frac{\partial}{\partial z}\} \\
&= -\frac{f'_u}{x}\mathcal{N}_f+\frac{1}{2x}\overline{\nabla}_{f'_u\partial x+\partial y}\{\frac{\partial}{\partial x}-f'_u\frac{\partial}{\partial y}-f'_v\frac{\partial}{\partial z}\} \\
&= -\frac{f'_u}{x}\mathcal{N}_f+\frac{1}{2x}\overline{\nabla}_{\partial y}\{\frac{\partial}{\partial x}-f'_u\frac{\partial}{\partial y}-f'_v\frac{\partial}{\partial z}\}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{f'_u}{x}\mathcal{N}_f-\frac{1}{2x}(f''_{y,u}\frac{\partial}{\partial y}+f''_{y,v}\frac{\partial}{\partial z}) \\
\overline{\nabla}_{\partial u}\mathcal{N}_f &= -\frac{f'_u}{x}\mathcal{N}_f-\frac{1}{2x}(f''_{y,u}\frac{\partial}{\partial u}+f''_{y,v}\frac{\partial}{\partial v}).
\end{aligned}$$

We conclude that $A_{\mathcal{N}}(\frac{\partial}{\partial u}) = \frac{1}{2x^2}\overset{\star}{A}_\xi(\frac{\partial}{\partial u}) = \frac{1}{2x}(f''_{y,u}\frac{\partial}{\partial u}+f''_{y,v}\frac{\partial}{\partial v})$

and $A_{\mathcal{N}}(\frac{\partial}{\partial v}) = \frac{1}{2x^2}\overset{\star}{A}_\xi(\frac{\partial}{\partial v}) = \frac{1}{2x}(f''_{z,u}\frac{\partial}{\partial u}+f''_{z,v}\frac{\partial}{\partial v})$

therefore: $\overset{\star}{A}_\xi = 2x^2 A_{\mathcal{N}}$ (conformal screen). (4.2.10)

$$\begin{aligned}
\eta &= \overline{g}(\mathcal{N}_f,\cdot) \\
&= \frac{1}{2x}\overline{g}(\frac{\partial}{\partial x}-f'_u\frac{\partial}{\partial y}-f'_v\frac{\partial}{\partial z},\cdot) \\
&= -\frac{1}{2x}(\mathrm{d}x+f'_u\mathrm{d}y+f'_v\mathrm{d}z) \\
&= -\frac{1}{2x}(f'_u\mathrm{du}+f'_v\mathrm{dv}+f'_u\mathrm{du}+f'_v\mathrm{dv}) \text{ since } \mathrm{d}x = f'_u\mathrm{du}+f'_v\mathrm{dv} \\
&= -\frac{1}{x}(f'_u\mathrm{du}+f'_v\mathrm{dv})
\end{aligned}$$

$\eta(\frac{\partial}{\partial u}) = \tau(\frac{\partial}{\partial u}) = -\frac{f'_u}{x}$ and $\eta(\frac{\partial}{\partial v}) = \tau(\frac{\partial}{\partial v}) = -\frac{f'_v}{x} \implies \tau = \eta$.

$$\begin{aligned}
\eta &= -\frac{1}{x}(f'_u \mathrm{d}u + f'_v \mathrm{d}v) \\
\mathrm{d}\eta &= \mathrm{d}(-\frac{1}{x}f'_u) \wedge \mathrm{d}u + \mathrm{d}(-\frac{1}{x}f'_v) \wedge \mathrm{d}v \\
&= (\mathrm{d}(-\frac{1}{x})f'_u - \frac{1}{x}\mathrm{d}f'_u) \wedge \mathrm{d}u + (\mathrm{d}(-\frac{1}{x})f'_v - \frac{1}{x}\mathrm{d}f'_v) \wedge \mathrm{d}v \\
&= \frac{f'_v f'_u}{x^2}\mathrm{d}v \wedge \mathrm{d}u - \frac{1}{x}f''_{v,u}\mathrm{d}v \wedge \mathrm{d}u + \frac{f'_v f'_u}{x^2}\mathrm{d}u \wedge \mathrm{d}v - \frac{1}{x}f''_{v,u}\mathrm{d}u \wedge \mathrm{d}v \\
&= (\frac{f'_v f'_u}{x^2} - \frac{1}{x}f''_{v,u})(\mathrm{d}v \wedge \mathrm{d}u + \mathrm{d}u \wedge \mathrm{d}v) \\
&= 0.
\end{aligned}$$

We take $\alpha = 2x^2$ and we have

$$\begin{aligned}
2\alpha\tau(\xi) + \mathrm{d}\alpha(\xi) &= 2\alpha + \mathrm{d}(2x^2)(\xi) \\
&= 2\alpha + 4x\mathrm{d}(x)(-x\{\frac{\partial}{\partial x} + f'_u\frac{\partial}{\partial y} + f'_v\frac{\partial}{\partial z}\}) \\
&= 2\alpha - 4x^2 = 0.
\end{aligned}$$

Now we need to check if α is constant on each leaf of the screen distribution. We have shown that $\overline{\nabla}_{\partial u}\mathcal{N} \in S(TM)$ since $dim\ S(TM) = 1 \implies S(TM) = \langle\overline{\nabla}_{\partial u}\mathcal{N}\rangle = \langle f''_{y,u}\frac{\partial}{\partial u} + f''_{y,v}\frac{\partial}{\partial v}\rangle$.

$$\begin{aligned}
\mathrm{d}\alpha(f''_{y,u}\frac{\partial}{\partial u} + f''_{y,v}\frac{\partial}{\partial v}) &= 4x\mathrm{d}x(f''_{y,u}\frac{\partial}{\partial u} + f''_{y,v}\frac{\partial}{\partial v}) \\
&= 4x(f'_u\mathrm{d}u + f'_v\mathrm{d}v)(f''_{y,u}\frac{\partial}{\partial u} + f''_{y,v}\frac{\partial}{\partial v}) \\
&= 4x\left(f'_u f''_{y,u} + f'_v f''_{y,v}\right) = 0 \text{ from Equation (4.1.9)}
\end{aligned}$$

so α is constant on each leaf of the screen distribution.

All the conditions of Theorem 4.2.8 are satisfied, so we conclude that the Levi-Civita connection of (M, g_α) and the induced connection on M coincide.

$$\nabla_\alpha = \nabla \text{ with } \alpha = 2x^2.$$

Example for which there is no coincidence between ∇_α and ∇.

4.2.12 Example. (Lightlike cone of $\mathbb{R}^3_1$)

Let us consider the lighlike cone of Example 4.1.7, and we endow it with the UCC-normalization rigging $N = \frac{1}{\sqrt{2}}\{-\frac{\partial}{\partial x} + \frac{y}{x}\frac{\partial}{\partial y} + \frac{z}{x}\frac{\partial}{\partial z}\}$ and with the corresponding rigged vector field $\xi = \frac{1}{\sqrt{2}}\{\frac{\partial}{\partial x} + \frac{y}{x}\frac{\partial}{\partial y} + \frac{z}{x}\frac{\partial}{\partial z}\}$.

The tangent space is spanned by $Y = \frac{\partial}{\partial y} + \frac{y}{x}\frac{\partial}{\partial x}$ and $Z = \frac{\partial}{\partial z} + \frac{z}{x}\frac{\partial}{\partial x}$.

$$\begin{aligned}
\overline{\nabla}_Y\xi &= \overline{\nabla}_Y\frac{1}{\sqrt{2}}\{\frac{\partial}{\partial x}+\frac{y}{x}\frac{\partial}{\partial y}+\frac{z}{x}\frac{\partial}{\partial z}\}\\
&= \frac{1}{\sqrt{2}}\overline{\nabla}_Y\{\frac{\partial}{\partial x}+\frac{y}{x}\frac{\partial}{\partial y}+\frac{z}{x}\frac{\partial}{\partial z}\}\\
&= \frac{1}{\sqrt{2}}\left[\frac{y}{x}(-\frac{y}{x^2}\frac{\partial}{\partial y}-\frac{z}{x^2}\frac{\partial}{\partial z})+\frac{1}{x}\frac{\partial}{\partial y}\right]\\
&= \frac{1}{\sqrt{2}x}\left[(1-\frac{y^2}{x^2})\frac{\partial}{\partial y}-\frac{yz}{x^2}\frac{\partial}{\partial z}\right].
\end{aligned}$$

We have

$$\begin{aligned}
\overline{g}((1-\frac{y^2}{x^2})\frac{\partial}{\partial y}-\frac{yz}{x^2}\frac{\partial}{\partial z},N) &= \frac{1}{\sqrt{2}}\overline{g}\left((1-\frac{y^2}{x^2})\frac{\partial}{\partial y}-\frac{yz}{x^2}\frac{\partial}{\partial z},-\frac{\partial}{\partial x}+\frac{y}{x}\frac{\partial}{\partial y}+\frac{z}{x}\frac{\partial}{\partial z}\right)\\
&= \frac{1}{\sqrt{2}}\left(\frac{y}{x}(1-\frac{y^2}{x^2})-\frac{y}{x}\frac{z^2}{x^2}\right)\\
&= \frac{y}{x\sqrt{2}}(1-\frac{y^2+z^2}{x^2})\\
&= 0 \implies (1-\frac{y^2}{x^2})\frac{\partial}{\partial y}-\frac{yz}{x^2}\frac{\partial}{\partial z}\in S(TM).
\end{aligned}$$

Therefore, since $\overline{\nabla}_Y\xi=-\tau(Y)\xi-\overset{\star}{A}_\xi(Y)$, we conclude that $\tau(Y)=0$ and $\overset{\star}{A}_\xi(Y)=-\frac{1}{x\sqrt{2}}\left[(1-\frac{y^2}{x^2})\frac{\partial}{\partial y}-\frac{yz}{x^2}\frac{\partial}{\partial z}\right]$ and by the same computation, we will obtain: $\tau(Z)=0$ and $\overset{\star}{A}_\xi(Z)=-\frac{1}{x\sqrt{2}}\left[(1-\frac{z^2}{x^2})\frac{\partial}{\partial z}-\frac{yz}{x^2}\frac{\partial}{\partial y}\right]$.

On the other hand:

$$\begin{aligned}
\overline{\nabla}_Y N &= \overline{\nabla}_Y\frac{1}{\sqrt{2}}\{-\frac{\partial}{\partial x}+\frac{y}{x}\frac{\partial}{\partial y}+\frac{z}{x}\frac{\partial}{\partial z}\}\\
&= \frac{1}{\sqrt{2}x}\left[(1-\frac{y^2}{x^2})\frac{\partial}{\partial y}-\frac{yz}{x^2}\frac{\partial}{\partial z}\right].
\end{aligned}$$

This implies that $A_N(Y)=\overset{\star}{A}_\xi(Y)$ and by the same $A_N(Z)=\overset{\star}{A}_\xi(Z)$, so $A_N=\overset{\star}{A}_\xi$

N is a closed rigging and For $\alpha=x$ which is constant along the leaf of the screen distribution, one has $\overset{\star}{A}_\xi\neq\alpha A_N$, so the Levi-Civita connection of (M,g_α) and the induced connection on M along the rigging N do not coincide.

$$\nabla_\alpha\neq\nabla,\quad\text{for}\quad\alpha=x. \tag{4.2.11}$$

We have used the following **SageMath** code to compute explicitly the connection coefficients and confirm the theoretical result obtained above.

```
M = Manifold(3, 'M'); M;X.<x, y, z> = M.chart(); X
```

$$M$$

$$(M,(x,y,z))$$

```
h = M.metric('h');h[0,0], h[1,1], h[2,2]=-1,1,1;h.disp();
```

$$h = -\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z$$

```
g = M.metric('g'); g[0,0], g[1,1], g[2,2],g[0,1] ,g[0,2] ,g[1,0] ,g[1,2] ,g[2,0] ,g[2,1]= -1+x,1+x,1+x,x,x,x,x,x,x;g.disp() #this is the metric g(alpha) where alpha=x.
```

evaluate

$$g = (x-1)\,\mathrm{d}x \otimes \mathrm{d}x + x\mathrm{d}x \otimes \mathrm{d}y + x\mathrm{d}x \otimes \mathrm{d}z + x\mathrm{d}y \otimes \mathrm{d}x + (x+1)\,\mathrm{d}y \otimes \mathrm{d}y + x\mathrm{d}y \otimes \mathrm{d}z + x\mathrm{d}z \otimes \mathrm{d}x + x\mathrm{d}z \otimes \mathrm{d}y + (x+1)\,\mathrm{d}z \otimes \mathrm{d}z$$

```
nabla = g.connection(); nabla
```

$$\nabla_g$$

```
XM= M.vector_field_module();# XM is the vector field modulus.
```

```
nab = h.connection(); nab;nab(h)==0# to check if the connection nab is h-metric.
```

$$\nabla_h$$

True

```
XM;N=XM([-1/(sqrt(2)),y/(sqrt(2)*x),z/(sqrt(2)*x)]);N.display()#XM.base_ring(),N is the rigging vector field.
```

$$\mathfrak{X}(M)$$

$$-\frac{1}{2}\sqrt{2}\frac{\partial}{\partial x} + \frac{\sqrt{2}y}{2\,x}\frac{\partial}{\partial y} + \frac{\sqrt{2}z}{2\,x}\frac{\partial}{\partial z}$$

```
xi=XM([1/(sqrt(2)),y/(sqrt(2)*x),z/(sqrt(2)*x)]);Y=XM([y/x,1,0]);Z=XM([z/x,0,1]);xi.disp() #xi is the rigged vector fied. Y and Z span the lightlike cone.
```

$$\frac{1}{2}\sqrt{2}\frac{\partial}{\partial x} + \frac{\sqrt{2}y}{2\,x}\frac{\partial}{\partial y} + \frac{\sqrt{2}z}{2\,x}\frac{\partial}{\partial z}$$

```
f=h(xi,xi);f.disp();#f=h(xi,Y);f.disp();f=h(xi,Z);f.disp()
```

$$\begin{array}{llcl} & M & \longrightarrow & \mathbb{R} \\ & (x,y,z) & \longmapsto & -\frac{x^2-y^2-z^2}{2\,x^2} \end{array}$$

```
t=nabla(g)#this is to check whether the connection is g-metric.
```

```
A=-nab(xi)(Y);A=-nab(N)(Y);A.display()#computing Astar(Y)=A_N(Y).
```

$$\left(-\frac{\sqrt{2}x^2 - \sqrt{2}y^2}{2\,x^3}\right)\frac{\partial}{\partial y} + \frac{\sqrt{2}yz}{2\,x^3}\frac{\partial}{\partial z}$$

```
P=nab(Y)(Z)#we want to compare the rigged connection nabla(alpha) and the induced connection nabla(N).
```

```
nablaN=P-g(A,Z)*N;nablaN.display()# Induced connection.
```

$$\left(-\frac{x^3 - xy^2 - yz^2 + \left(x^2 - (x-1)y - y^2\right)z}{2\,x^3}\right)\frac{\partial}{\partial x} + \left(\frac{x^3y - xy^3 - y^2z^2 + \left(x^2y - (x+1)y^2 - y^3\right)z}{2\,x^4}\right)\frac{\partial}{\partial y} + \left(-\frac{yz^3 - \left(x^2 - (x+1)y - y^2\right)z^2 - \left(x^3 - xy^2\right)z}{2\,x^4}\right)\frac{\partial}{\partial z}$$

```
Q=nabla(Y)(Z);Q.display()# Levi-Civita connection.
```

$$\left(\frac{2\,x^4 + 2\,x^3y + x^3 + \left(2\,x^3 + \left(2\,x^2 - 3\,x - 2\right)y\right)z}{2\,(x^4 + x^3)}\right)\frac{\partial}{\partial x} + \left(-\frac{x^3 + \left(x^2 - x\right)y + \left(x^2 + (x-2)y - x\right)z}{2\,(x^3 + x^2)}\right)\frac{\partial}{\partial y} + \left(-\frac{x^3 + \left(x^2 - x\right)y + \left(x^2 + (x-2)y - x\right)z}{2\,(x^3 + x^2)}\right)\frac{\partial}{\partial z}$$

```
nablaN==Q # Do the induced and the Levi-Civita connections coincide ?
```

False

5. Conclusion

In this work, we used notions of screen distribution, rigging and rigged vector field to study the geometry of lightlike hypersurfaces. We recalled the Gauss and Weingarten formulas in the case of semi-Riemannian and lightlike hypersurfaces and gave the Gauss-Codazzi equations for both cases. In our main work, we stated and proved the sufficient and necessary conditions for the coincidence in the case $\alpha = 1$ [see [2]]. Later on, we stated and proved the necessary conditions for coincidence in the case where α is a non vanishing function [see [4]]. Furthermore, we proved that if the lightlike hypersurface is totally geodesic and the rotation function τ is such that $\mathrm{d}\tau \neq 0$ then the induced and the rigged connections never coincide for any change of rigging and later we proved that if (N, α) is a solution for the coincidence, then $(\psi N, \frac{\alpha}{\psi^2})$ does not change the associated metric hence is also a solution for coincidence and the latter allowed us to remark that we can have coincidence even when the rigging vector field is not close or when the function α is not constant along leaves of the screen distribution.

For further researches, we will be considering the case where g_α is a Riemannian metric induced on a null hypersurface and observe which relation is between the induced geometric objects on M by the ambient and the geometric objects of the couple (M, g_α).

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