Riemannian Geometry

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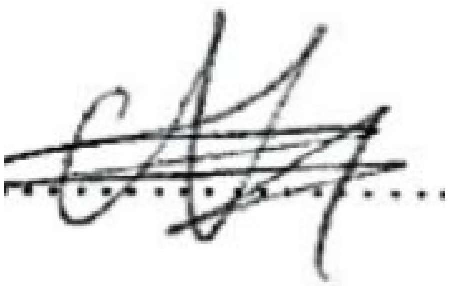
Abstract

In this work we are going to provide some basic concepts in Riemannian Geometry, we will see how to define a metric on a differentiable manifold, that we call a Riemannian metric, and thus define a structure of a Riemannian manifold. Having done so, we introduce the notion of covariant derivative, to talk about one of the fundamental theorems of Riemannian geometry that shows the existence of a Levi-Civita connection which is useful to develop two fundamental notions of Riemannian Geometry: geodesics and curvature. We will present numerous examples.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



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1. Introduction

Introduced by **Bernhard Riemann** around 1854 [7], Riemannian geometry is a branch of differential geometry. Riemann developed the fundamental notions of geometric shape and curvature in order to generalize the traditional geometry that was limited to the Euclidean space of dimension 3. This field underwent several evolutions due to the hard work of mathematicians like **Greogorio Ricci-Curbastro** [9], **Tullio Levi-Civita** [1] etc. and in the mid-1930's this research was concluded by the establishment of the Whitney plunge Theorem in 1936, that allowed to define a formal map of the Riemannian geometry.

The basic objects of Riemannian geometry are gradually finding many applications also outside of Riemannian geometry itself, for instance, in the study of metric spaces. **Gromov** for example, defines notions in geometric group theory, such as the hyperboloic group. In the same way, **Villani** [3], **Lott** [6] and **Sturm** [11] introduced in 2010 an extented "synthetic" vision of the notion of Ricci curvature minus a formulation of optimal transport on a metric space with a measure.

The main characters of differential geometry are objects called manifold, they can be seen locally as an n-dimensional subspace of the Euclidean space, we define a differentiable structure on them, to turn them into differentiable manifolds. A Riemannian manifold, the main focus of Riemannian geometry, is a differentiable manifold endowed with a Riemannian metric. This metric is a tool which is useful to calculate distances, measure angles, evaluate volume and so on.

In recent years, many notions of Riemannian geometry have proven to be useful in different research areas: in graph theory, for instance, where geodesics are very useful to determine the shortest path between two vertexes, in neural networks and so on.

The main objective of this thesis is to study the main basic concepts in Riemannian geometry focusing in particular on the two fundamentals notions of geodesic and curvature, with many examples.

This thesis is structured in the following way: In Chapter 2, we start introducing some basic notions in differential geometry. In Chapter 3, we introduce the notion of Riemannian metric and we then use it to define Riemannian manifolds and the Levi-Civita connection. We conclude presenting geodesics. In Chapter 4, we discuss the notion of curvature. We conclude with a summary of the results.

2. Preliminary Results

2.1 Manifolds

Before defining the notion of an n-dimensional manifold M, we need to introduce some key notions to have a better understanding for what we are going to do.

2.1.1 Definition. Given a topological space M, we will say that it is a **Hausdorff space** if given two points p_1, p_2 in M, we can find two open sets U_1 and U_2 such that $p_i \in U_i, i = 1, 2,$ and $U_1 \cap U_2 = \emptyset$.

2.1.2 Definition. A **n-dimensional chart** of M is the couple (U, φ), where U is and open set in the topological space M and φ is a homeomorphism $\varphi : U \longrightarrow \varphi(U) \subset \mathbb{R}^n$.

2.1.3 Examples.

- The couple $(\mathbb{R}^n, id_{\mathbb{R}^n})$ is a chart of $\mathbb{R}^n$ at each points.
- Let U be an open set of $\mathbb{R}^n$, and $j : U \hookrightarrow \mathbb{R}^n$ an injection function, then (U, j) is a chart for all points in U.

2.1.4 Definition. Given two charts (U, φ) and (V, ψ) such that $U \cap V \neq \emptyset$, the **transition function** is the map $\varphi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \varphi(U \cap V)$. If this map is a homeomorphism we will say that the two charts are **compatible**. See figure in the Appendix A.1b

2.1.5 Definition. An **atlas** of M is a family $\{(U_i, \varphi_i)\}_{i \in I}$ of charts, such that $(U_i)_{i \in I}$ defines a cover of M, and any two charts (U_i, φ_i) and (U_j, φ_j) for $i, j \in I$ $(i \neq j)$ are compatibles.

Now what is an n-dimensional manifold?

2.1.6 Definition. A topological space M is called **n-dimensional manifold**, if the following three conditions are satisfied:

1. M is a Hausdorff space;
2. For any $p \in M$, there exists an open subset U that contains p, which is homeomorphic to an open subset $V \subset \mathbb{R}^n$. This means that we can find $\psi : U \longrightarrow V$ such that ψ is a homeomorphism;
3. M has a countable basis of open sets.

From Definition 2.1.6 it turns out that a manifold is locally connected , locally compact due to condition 2, it is also the union of countably many compact subsets. The local coordinates around a point $p \in M$, is defined by taking a chart (U_i, φ_i) coming from an atlas of M, such that U_i is an open set that contains p and $\varphi_i : U_i \longrightarrow \mathbb{R}^n$ homeomorphism, $\varphi_i(p) \in \mathbb{R}^n$. This means that $\varphi_i(p) = (x_i^1(p), \cdots, x_i^n(p))$, with $x_i^k : U_i \longrightarrow \mathbb{R}$ are differentiable functions, so that the family $(x_i^1, \cdots, x_i^n)$ is the local frame in the chart (U_i, φ_i).

2.1.7 Examples.

1. if we take the atlas $(\mathbb{R}^n, Id_{\mathbb{R}^n})$, then $\mathbb{R}^n$ endowed with its usual topology, $(\mathbb{R}^n, Id_{\mathbb{R}^n})$ is an n-dimensional manifold.

2. The unit sphere S^n of $\mathbb{R}^{n+1}$, is homeomorphic to $\mathbb{R}^n$. So it is a topological space which is separable. Next we define our atlas by $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ with $U_1 = S^n - \{N\}$ and $U_2 = S^n - \{S\}$, where we denote by $N = (0, 0, \cdots, 1)$ the North point and by $S = (0, 0, \cdots, -1)$ the south point. we define by φ_1 the stereographic projection into the intersection of the hyperplane $x_{n+1} = 0$, and the line that passes through p and N. We denote by φ_2 the stereographic projection onto the intersection of the hyperplane $x_{n+1} = 0$, the line that passes through $p \in S^n$ and S. one can show that $S^n = U_1 \cup U_2$. Let any $p \in S^n$, then $p \in U_1$ or $p \in U_2$, and according to the open in which belongs, p we take the stereographic projection φ_1 or φ_2 which is an homeomorphism.

Knowing that M is a topological space and separable in the sense of Hausdorff and has a basis of countable open sets, we want now to define a structure of differentiable manifold on M.

2.2 Differentiable manifolds and tangent space

We start by recalling the definition of regular surface that we will then generalize to the concept of differentiable manifold. Let $S \subset \mathbb{R}^3$, S is a **regular surface** if for all $p \in S$, there exists an open set V that contains p and $h:\ U \subset \mathbb{R}^2 \longrightarrow V \cap S$, such that:

- h is a diffeomorphism.
- The differential of h at a point q dh_q: $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is injective for all $q \in U$.

2.2.1 Definition. A **differentiable manifold** of dimension n, is a set M, with a family of injective mapping $X_\alpha :\ \mathbb{R}^n \supset U_\alpha \longrightarrow M$, such that:

1. $\cup_{\alpha \in I} X_\alpha(U_\alpha) = M$.
2. Given two mappings such that $X_\alpha(U_\alpha) \cap X_\beta(U_\beta) \neq \emptyset$, the transition map $X_\beta \circ X_\alpha^{-1}$ is a differentiable function.
3. $\{(U_\alpha, X_\alpha)\}_{\alpha \in I}$ is a maximal atlas(see Proposition 6.5 on [5]).

Hence every family $\{(U_\alpha, X_\alpha)\}_{\alpha \in I}$ that satisfies 1 and 2 of the Definition 2.2.1 is called differentiable structure on M.

2.2.2 Example.

1. $M = \mathbb{R}^n$

   Considering the trivial atlas $\{(\mathbb{R}^n, Id_{\mathbb{R}^n})\}$, hence $\mathbb{R}^n$ has a structure of differential manifold.

2. $S^n = \{(x_1, ..., x_{n+1}) | \sum_{i=1}^{n+1} x_i^2 = 1\} \subset \mathbb{R}^{n+1}$.

as in example 2.1.7 consider the two stereographic projections, to be in phase with the definition of differential manifold let us define the atlas. The stereographic projection is defined by the following formula:

$$\varphi_1(x_1, x_2, ..., x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, ..., x_n)$$

Similar formula holds for φ_2 . Note that φ_1 is differentiable, injective and maps $S^n - \{N\}$ to $\mathbb{R}^n$. The same property holds for φ_2. Hence φ_1 and φ_2 are invertible. Thus the set $\{(\mathbb{R}^n, \varphi_1^{-1}), (\mathbb{R}^n, \varphi_2^{-1})\}$ defines a differentiable structure on S^n.

In what follows, we will denote by M^n a n-dimensional differentiable manifold. The notion of equivalence for two differentiable manifolds is the notion of diffeomorphism between them, the question is how can we define differentiable maps between two differentiable manifolds?

2.2.3 Definition (differentiable maps between two manifolds). Let M_1^n and M_2^m two differentiable manifolds. Let us consider the map $f : M_1 \longrightarrow M_2$, and $p \in M_1$

- f is **differentiable** at a point p if there exists a chart (U, φ) of M_1 at p, and a chart (V, ϕ) of N at $f(p)$ such that $f(U) \subset V$ and the map $\phi \circ f \circ \varphi^{-1} : \varphi(U) \longrightarrow \phi(V)$ is differentiable at $\varphi(p)$.
- f is **differentiable on** M_1 if it is the case at each point of M_1.

2.2.4 Remark.

- The function $\phi \circ f \circ \varphi^{-1}$ is a local expression for f in relation to the local charts (U, φ) and (V, ϕ).
- this local expression does depend on the choice of the two charts.

2.2.5 Definition. The **tangent space** at a point $p \in M^n$, denoted by T_pM, is defined as follows:

$$T_pM = \{v : \ \mathcal{D}(M) \longrightarrow \mathbb{R} |\ \exists \gamma : I \subset \mathbb{R} \longrightarrow M\}$$

where I is an interval of $\mathbb{R}$ which contains the origin 0, and γ is a differentiable curve such that $\gamma(0) = p$ and $\gamma'(0) = v$ and satisfies

$$\gamma'(0)f = \frac{d(f \circ \gamma)}{dt}|_{t=0}.$$

Let $\mathcal{D}(M)$ the set of differentiable functions defined on M. Take a parametrization $X : \ U \longrightarrow M^n$, such that $X(0) = p$, and $f \in \mathcal{D}(M)$ defined in the neighborhood of p, and consider $\gamma : \ I \ni 0 \longrightarrow M$. In this parametrization we have:

$$f \circ X(q) = f(x_1, ..., x_n), \qquad q = (x_1, ..., x_n) \in U$$

and

$$X^{-1} \circ \gamma(t) = (x_1(t), ..., x_n(t))$$

Indeed

$$\gamma'(0)f = \frac{d(f \circ \gamma)}{dt}\Big|_{t=0} \quad \text{this implies} \quad \gamma'(0)f = \frac{df(x_1(t), ..., x_n(t))}{dt}$$

$$\gamma'(0)f = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f \frac{\partial x_i(t)}{dt}|_{t=0}$$

$$\gamma'(0)f = \left(\sum_{i=1}^{n} x_i'(0) \left(\frac{\partial}{\partial x_i} \right)_{|t=0} \right) f \quad \forall f \in \mathcal{D}(M);$$

Then $\gamma'(0) = \sum_{i=1}^{n} x_i'(0) \left(\frac{\partial}{\partial x_i}\right)_{|t=0}$, we can conclude that the basis of T_pM is given by $\left(\frac{\partial}{\partial x_i}\right)$.

2.2.6 Remarks.

- The point p of M is called the origin of all the tangent vectors at p.
- Let M be a differentiable manifold of dimension n. For all $p \in M$, T_pM is a real vector space of dimension n.

2.2.7 Definition. Let M^m and N^n be two differentiable manifolds and $f: M \longrightarrow N$ a diffeomorphism. We define the **differential map** of f at p, like a linear map denoted by df_p, defined as follows:

$$\begin{array}{rcl} df_p : T_pM & \longrightarrow & T_{f(p)}N \\ u & \longrightarrow & df_p(u) \end{array}$$

with $df_p(u) = \frac{d}{dt} f(\gamma(t))\big|_{t=0}$, where $\gamma(0) = p$ and $\dot{\gamma}(0) = u$.
This notion of differential map is very useful to define the notions immersion, and embedding.

2.2.8 Definition. Let M^m, N^n two differentiable manifolds, $f: M \longrightarrow N$, a differentiable map of classe C^r.

- f is an **immersion** if and only if the differential map df_p is injective for all $p \in M$
- f is an **embedding** if and only is an immersion that in addition is an homeomorphism defined from M into $f(M)$.

2.3 Vector Bundles

The aim of this section is to introduce the notion of vector bundles. Given a differential manifold M^n, take $p \in M$ and define the tangent space at p by T_pM. If we consider the union of all the tangents spaces defined for all the points in M, we obtain the **tangent bundle** denoted by $TM = \cup_{p\in M}(T_pM)$. An element of TM is a couple (p, v) such that $p \in M$ and $v \in T_pM$. This notion of tangent bundle will be useful to define the vector fields on the differentiable manifold M. Consider M and E two differentiable manifolds, and $\Pi : E \longrightarrow M$ a surjective projection of class C^∞.

2.3.1 Definition. E is a real **vector bundle** of rank n over M when:

1. For all $p \in M$, $E_p := \Pi^{-1}(p)$ has a structure of real vector space of dimension n.
2. Local trivialization.

   For all $p \in M$, there exists an open neighborhood U of p and a diffeomorphism

   $$\phi : \Pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$$

   such that the following diagram commutes:

   $$\begin{tikzcd} & U & \\ \Pi^{-1}(U) \arrow[ur, "\Pi"] \arrow[rr, "\phi"'] & & U \times \mathbb{R}^n \arrow[ul, "Pr_1"'] \end{tikzcd}$$

   The restriction of ϕ to E_p defined by $\phi|_{E_p} : E_p \longrightarrow \{p\} \times \mathbb{R}^n$ is a linear isometric.

2.3.2 Remarks.

- In this Definition 2.3.1 for all $p \in M$, $E_p \cong \mathbf{V}$, where $\mathbf{V}$ is a real vector space of dimension n.
- We write $(E, M, \Pi, \mathbf{V})$ to denote that E is a real vector bundle over M.
- M is the base space and E total space. For this Definition 2.3.1 $\mathbf{V} = \mathbb{R}^n$.

Let $(E, M, \Pi, \mathbf{V})$ be a vector bundle, and U an open subset of M. A **local section** C^∞, s of E over U is an application defined by: $s : U \longrightarrow E$ such that $\Pi \circ s = id_u$. Knowing that a differentiable manifold M^n induces a differentiable manifold TM. In Definition 2.3.1 for $E = TM$, we have: $\Pi : TM \longrightarrow M$, that assigns $(p, v) \in TM$ to $p \in M$. Π is a subjective projection C^∞. Let us show that $(TM, M, \Pi, \mathbb{R}^n)$ is a real vector bundle.

1. Let $p \in M$ we have:

$$\begin{aligned} E_p = \Pi^{-1}(p) = \{(q,v) \in TM \big|\ \Pi(q,v) = p\} \quad &\Longrightarrow \quad E_p = \{(q,v) \in TM|\ q = p\} = \{p\} \times T_pM \\ \text{Since } \ T_pM \cong \mathbb{R}^n \quad & \\ &\Longrightarrow \quad E_p = \{p\} \times \mathbb{R}^n \cong \mathbb{R}^n. \end{aligned}$$

   Then E_p is a real vector space of dimension n.

2. Local trivialization

   Let $p \in M$, there exists a chart(U, ψ) coming from an atlas of M such that $U \ni p$, this implies $(T\psi, \psi(U) \times \mathbb{R}^n)$ is a chart of TM, where

$$\begin{array}{rcl} T\psi :\ \Pi^{-1}(u) & \longrightarrow & \psi(U) \times \mathbb{R}^n \\ \left(q, v^i \dfrac{\partial}{\partial x_i}|_q\right) & \longrightarrow & (\psi(U), v^1, ..., v^n); \end{array} \tag{2.3.1}$$

   and $(x^1, ..., x^n)$ is the local coordinates in the chart (U, ψ). Let us define ϕ. Having $T\psi$ and $\psi^{-1} \times id_{\mathbb{R}^n} : \psi(U) \times \mathbb{R}^n \longrightarrow U \times \mathbb{R}^n$; we can write: $\phi = (\psi^{-1} \times id_{\mathbb{R}^n}) \circ T\psi$.

   Let us show now the commutativity. We need to show that $Pr_1 \circ \phi = \Pi$. Let $\left(q, v^i \frac{\partial}{\partial x_i}|_q\right)$ in $\Pi^{-1}(U)$, we have:

$$\begin{aligned} Pr_1 \circ \phi \left(q, v^i \frac{\partial}{\partial x_i}|_q\right) &= Pr_1 \circ \left((\psi^{-1} \times id_{\mathbb{R}^n}) \circ T\psi\right) \left(q, v^i \frac{\partial}{\partial x_i}|_q\right) \\ &= Pr_1 \left(\psi^{-1}(\psi(q)), id_{\mathbb{R}^n}(v^1, ..., v^n)\right) \\ &= Pr_1 \left(q, v^1, ..., v^n\right) = q \\ Pr_1 \circ \phi \left(q, v^i \frac{\partial}{\partial x_i}|_q\right) &= \Pi \left(q, v^i \frac{\partial}{\partial x_i}|_q\right) \implies Pr_1 \circ \phi = \Pi \end{aligned}$$

   So it commutes.

   Now let us show that $\phi|_{E_p} :\ \Pi^{-1}(q) \longrightarrow \{q\} \times \mathbb{R}^n$ is a linear isomorphism. We have:

$$\phi \left(q, v^i \frac{\partial}{\partial x_i}|_q\right) = (q, (v^1, ..., v^n))$$

   So $\phi|_{E_p}$ is a linear isomorphism. Which ends the local trivialization.

We can conclude that $(TM, M, \Pi, \mathbb{R}^n)$ is a vector bundle of TM over M. Its global sections are defined as follows:

$$\begin{aligned} s : M &\longrightarrow TM \\ p &\longrightarrow (p, s(p)) \text{ such that } s(p) \in T_pM \end{aligned}$$

This is because we have $\Pi \circ s(p) = \Pi(p, s(p)) = p$ ie $\Pi \circ s = id_M$. Therefore sections of TM above M are **vector fields**. Hence this gives rise to the following definition.

2.3.3 Definition. A **vector field** is a map $X : M \longrightarrow TM$, which associates to a point $p \in M$ an element $X(p) \in T_pM$, such that: $\Pi \circ X = id_M$.

2.3.4 Remark.

1. The vector field is said to be differentiable if X is differentiable.

2. Defining $\mathcal{F}(M)$ to be the set of function defined on M, a vector field can also be viewed as a map $X : \mathcal{D}(M) \longrightarrow \mathcal{F}(M)$ which maps $f \in \mathcal{D}(M)$ to $Xf \in \mathcal{F}(M)$. Considering a local chart (U, ρ) of a point $p \in M$, with is local coordinates $(x_1, ..., x_n)$ we have:

$$(Xf)(p) = a^i(p) \frac{\partial f}{\partial x_i}(p);$$

   where $a^i : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ are differentiable functions in sense to make X to be differentiable.

3. Let $f, g \in \mathcal{D}(M)$ we have: $X(fg) = X(f)g + fX(g)$.

2.3.5 Definition. Let X and Y be two differentiable vector fields. The **lie bracket** of X and Y is denoted by $[X, Y]$ and is defined as follows: $[X, Y] = XY - YX$

This lie bracket has the following properties. For any three vector fields $X,\ Y,\ Z$ and $f, g \in \mathcal{D}(M)$ we have:

1. $[X, Y] = -[Y, X]$ (antisymmetry),
2. $[aX + bY, Z] = a[X, Z] + b[Y, Z],\ a, b \in \mathbb{R}$(linearity),
3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity),
4. $[X, Y](fg) = f[X, Y](g) + [X, Y](f)g$,
5. $[X, gY] = [X, Y]g + X(g)Y$.

3. Riemannian manifolds and Levi-Civita connection

3.1 Riemannian metrics

3.1.1 Definition. A **Riemannian metric** on a differentiable manifold M is a correspondence which associates to each point p of M an inner product $<,>_p$ on the tangent space T_pM. It is Differentiable in the sense that given a local coordinate around p, $X : U \subset \mathbb{R}^n \longrightarrow M$, with $X(x^1, ..., x^n) = q \in X(U)$ and $\frac{\partial}{\partial x_i}(q) = dX_q(0, ..., 1, 0, ..., 0)$ with 1 on the i^{th} position, then:

$$< \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) >= g_{ij}(q) \tag{3.1.1}$$

The g_{ij} in (3.1.1) defines differentiable functions. We have $g_{ij} = g_{ji}$ due to the symmetry of the inner product in the coordinate X. A **Riemannian manifold** is a differentiable manifold, equipped with a Riemannian metric.

Since two differentiable manifolds M^m and N^n, are equivalent if we can define a diffeomorphism between them, this gives rise to the question: How can we say that two Riemannian manifolds are equivalents?.

3.1.2 Definition. Let M and N be two Riemannian manifolds, and $f : M \longrightarrow N$ a diffeomorphism. The map f is an **isometry** if it preserves the inner product:

$$< u, v >_p =< df_p(u), df_p(v) >_{f(p)}, \ \forall p \in M, \ u, v \in T_pM \tag{3.1.2}$$

Hence two Riemannian manifolds; M and N, are **locally isometric** at p; if we can find an open neighborhood $U \subset M$ which contains p and $f : U \longrightarrow f(U)$ is a diffeomorphism; that satisfies (3.1.2).

3.1.3 Some Examples of Riemannian manifolds.

1. Euclidian space $M = \mathbb{R}^n$,

   We know that $(\mathbb{R}^n, Id)$ is a differential manifold, and since we have $T_pM \cong \mathbb{R}^n$, $\forall p \in \mathbb{R}^n$, we can define the usual inner product $< u, v >_p = \sum_{i=1}^{n} u_i v_i$, where $u = (u_1, ..., u_n), v = (v_1, ..., v_n) \in T_pM$. Then M with this metric, that we denote by $(M, <, >)$, is a Riemannian manifold.

2. Immersed manifolds.

   Let us consider two differentiable manifolds M^n, N^{n+k},$n, k \in \mathbb{N}^*$ and $f : M^n \longrightarrow N^{n+k}$, an immersion. If N is a Riemannian manifold, then M is a Riemannian manifold equipped with the inner product defined by :

   $$< u, v >_p =< df_p(u), df_p(v) >_{f(p)} . \tag{3.1.3}$$

   Since M^n is a differentiable manifold, we just need to prove that $<, >_p$, defined in (3.1.3), is an inner product on T_pM

   - It is evident that $<, >_p$ is bilinear and symmetric due to the fact that $<, >_{f(p)}$ is an inner product on N.

- Positive definite. Let $u \in T_pM$, let us prove that $<,>$ is Positive definite, which means that

$$< u, u >_p \geq 0 \tag{3.1.4}$$

and

$$< u, u >_p = 0 \implies u = 0_{\mathbb{R}^n}. \tag{3.1.5}$$

We have:

$$\begin{aligned} < u, u >_p = 0 &\implies < df_p(u), df_p(u) >_{f(p)} = 0 \\ &\implies df_p(u) = 0 \text{ because } <,>_{f(p)} \text{ is an inner product on } T_{f(p)}N \\ &\implies u \in \ker df_p \\ &\implies u = 0_{\mathbb{R}^n} \text{ since } df_p \text{ is injective} \end{aligned}$$

which prove (3.1.5). By definition we have

$$< u, u >_p \; = \; < df_p u, df_p u >_{f(p)} \geq 0 \text{ , because } <,>_{f(p)} \text{ is an inner product on } T_{f(p)}N$$

Which prove (3.1.4)

So we can conclude that $<,>$ is an inner product on T_pM, and then M equipped with this metric is a Riemannian manifold.

3. The unit sphere S^{n-1} of $\mathbb{R}^n$, $n \geq 2$

   Let us prove that it is a Riemannian manifold with a metric that we are going to define on it. S^{n-1} is an embedded manifold because it is the pre-image of a regular value by the norm square, hence it is a Riemannian manifold, equipped with the metric induced by $\mathbb{R}^n$ on S^{n-1}.

3.1.4 Proposition. Given a differentiable manifold M, we can equip it with a Riemannian metric.

Proof. The proof can be found in Do Carmo's Book [2] □

Up to now we have seen how we can define a Riemannian metric on a differentiable manifold M to make it a Riemannian manifold then how can we calculate the length of a curve over M?. Let $c : I \subset \mathbb{R} \longrightarrow M$ be a differentiable map, which means that c is a curve on M.

3.1.5 Definition. A vector field along a curve c, is a tangent vector on c; that we can see as a map $V : I \longrightarrow TM$, such that the folllowing diagram commutes:

$$\begin{tikzcd} & I \arrow[dl, "c"'] \arrow[dr, "V"] & \\ M & & TM \arrow[ll, "\Pi"] \end{tikzcd}$$

ie: $\Pi \circ V = c$

Now given a segment on c, which is a restriction of c to a closed interval $[a, b] \subseteq I; a, b \in \mathbb{R}(a \leq b)$, and a Riemannian manifold M. we define the **length** of the curve c along the segment $[a, b]$, denoted by $l_a^b(c)$ as follows:

$$l_a^b(c) = \int_a^b (< \frac{dc}{dt}, \frac{dc}{dt} >)^{\frac{1}{2}} dt. \tag{3.1.6}$$

To conclude this section, we can resume that given a differentiable manifold M, we can always define a Riemannian metric on it by using the Proposition **2.1.5**, and then define at each point $p \in M$ a measurement of the length of the tangent vector, also the angles between them. In the following section we are going to adopt the following notation:

$$\begin{array}{rcl} \mathfrak{X}(M) & = & \text{Set of all vector fields of class } C^\infty \text{ on } M \\ \mathcal{D}(M) & = & \text{the ring of real valued function } C^\infty \text{ on } M \end{array}$$

In what follows we will work with an $n-$dimensional Riemannian manifold M, and its metric $<,>$ that we will denote by $(M^n, <,>)$, $n \geq 2$.

3.2 Connection and covariant derivative

3.2.1 Definition. Given a Riemannian manifold M;a **connection** ∇ is a correspondence defined as follows:

$$\begin{array}{rcl} \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(M) \\ (X, Y) & \longrightarrow & \nabla_X Y \end{array} \tag{3.2.1}$$

with the following properties:

i) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$

ii) $\nabla_{(fX+gZ)}Y = f\nabla_X Y + g\nabla_Z Y$

iii) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$

$\forall \;\; X, Y, Z \in \mathfrak{X}(M), f, g \in \mathcal{D}(M).$

In this definition, the connection $\nabla_X Y$ is the derivative of Y in the direction of X .

3.2.2 Remark. From the Definition 3.2.1

1. The second property satisfied, means that the connection ∇ is linear in X on the space $\mathfrak{X}(M)$.
2. The third one shows us that with respect to Y, it acts like a derivation.

The usual derivative $\frac{dv(t)}{dt}$ of a vector field along a curve $v(t)$ does not belong to TM in general, hence the notion $\frac{d}{dt}$ is not an intrinsic property. To avoid this, we introduce the notion of the covariant derivative (that we can geometrically see like the orthogonal projection of the usual derivative of $v(t)$ over TM).

3.2.3 Definition. A **Covariant-derivative** is a correspondence D, which associates to a vector field V along $c : I \subset \mathbb{R} \longrightarrow M$ (curve on M) a vector field $\frac{DV}{dt}$ on TM, such that:

- $\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt}$,
- $\frac{D(fV)}{dt} = \frac{Df}{dt}V + f\frac{DV}{dt}$,

- Given V vector feild along c, such that it induce by $Y \in Vect(M)$, which means that $\exists t \in I$ such that $V(t) = Y(c(t))$, we have :

$$\frac{DV}{dt} = \frac{DY(c(t))}{dt} = \nabla_{\frac{dc(t)}{dt}} Y. \tag{3.2.2}$$

This last property makes sense, since $\nabla_X Y(p)$ depends on the value of $X(p)$ and the value of Y along the curve, tangent to X at p .This last property shows us also that the connection allows us to differentiate a vector along a curve.

3.2.4 Lemma. The connection is an intrinsic property, it depends on the ambient space.

Consider the local coordinate around a point $p \in M$, $p = (x_1, ..., x_n)$, the Components of connection can be express in terms of Christoffel symbols:

$$\nabla_{X_i} X_j = \Gamma_{ij}^k X_k. \tag{3.2.3}$$

where Γ_{ij}^k are differentiable functions.
For the proof of Lemma 3.2.4 see Appendix A.0.4. Having the Definition 3.2.3, we can define some key concepts of parallel vector, and parallel transport.

3.2.5 Definition. Let $(M, <,>)$ be a Riemannian manifold, with its connection ∇; a vector field V along the curve c, such that $\frac{DV}{dt} \equiv 0 \quad \forall t \in I$ is called a **parallel vector** along c.

3.2.6 Lemma. Let $M^2 \subset \mathbb{R}^3$ be a surface in $\mathbb{R}^3$ with the induced Riemannian metric. Let $c : I \longrightarrow M$ a differential curve on M and V a vector field tangent to M along c. Then V is parallel iff $\frac{dV}{dt}$ is perpendicular to $T_{c(t)}M \subset \mathbb{R}^3$.

Proof. Suppose that V is parallel. Let $\omega \in T_{c(t)}M$, consider the orthogonal projection of $\frac{dV}{dt}$ on $T_{c(t)}M$, we know that $\frac{dV}{dt} \in T_{c(t)}M \bigoplus (T_{c(t)}M)^{\perp}$; then $\frac{dV}{dt} = \left(\frac{dV}{dt}\right)^{\top} + \left(\frac{dV}{dt}\right)^{\perp}$; where $\left(\frac{dV}{dt}\right)^{\top} \in T_{c(t)}M$ and $\left(\frac{dV}{dt}\right)^{\perp} \in (T_{c(t)}M)^{\perp}$. <u>**Recall**</u> The covariant derivative is just the projection of the usual derivative in the ambient space. In this case the ambient space is $T_{c(t)}M$, the covariant derivative is $\left(\frac{dV}{dt}\right)^{\top}$

We have:

$$\begin{aligned}
< \frac{dV}{dt}, \omega >=< \left(\frac{dV}{dt}\right)^{\top} + \left(\frac{dV}{dt}\right)^{\perp}, \omega > \iff & < \frac{dV}{dt}, \omega >=< \left(\frac{dV}{dt}\right)^{\top} + \left(\frac{dV}{dt}\right)^{\perp}, \omega > \\
\iff & < \frac{dV}{dt}, \omega >=< \left(\frac{dV}{dt}\right)^{\top}, \omega > + < \left(\frac{dV}{dt}\right)^{\perp}, \omega >
\end{aligned}$$

Since $< \left(\frac{dV}{dt}\right)^{\perp}, \omega >= 0$, because $\left(\frac{dV}{dt}\right)^{\perp} \in (T_{c(t)}M)^{\perp}$

$$\begin{aligned}
< \left(\frac{dV}{dt}\right)^{\top} + \left(\frac{dV}{dt}\right)^{\perp}, \omega > \iff & < \frac{dV}{dt}, \omega >=< \left(\frac{dV}{dt}\right)^{\top}, \omega > \\
& \text{by hypothesis } \left(\frac{dV}{dt}\right)^{\top} = \frac{DV}{dt} = 0 \\
\iff & < \frac{dV}{dt}, \omega >= 0.
\end{aligned}$$

$\square$

3.2.7 Example. Let us consider $S^2 \subseteq \mathbb{R}^3$ the sphere of $\mathbb{R}^3$; with the metric induced by the metric of $\mathbb{R}^3$ on S^2. The velocity field along great circles, parametrized by arc length, is a parallel field. How to see it? Consider the parametrization of the great circle define by

$$\begin{aligned} \gamma : [0; 2\pi] &\longrightarrow S^2 \\ t &\longrightarrow (cost, sint, 0). \end{aligned} \tag{3.2.4}$$

The velocity is given by $V(t) = \dot{\gamma}(t) = (-sint, cost, 0)$, hence $\frac{dV}{dt} = \ddot{\gamma}(t) \in (T_{\gamma(t)}S^2)^{\perp}$, so it is orthogonal to $T_{\gamma(t)}S^2$. By Lemma 3.2.6, we conclude that $V(t)$ is a parallel field. For the sketch see in the Appendix figure A.1a, our curve is in red and the parallel field is arrow.

The definition of a parallel vector transport comes directly, from the following proposition.

3.2.8 Proposition. Consider a Riemannian metric $(M^n, <,>)$, and a connection ∇. Taking $V_0 \in T_{c(t_0)}M$; there exists a unique parallel vector along c. V; such that $V(t_0) = V_0$.

Proof. Confer Do Carmo [2] □

This unique vector field defined in Proposition 3.2.8 is called the **parallel transport** along the curve c; which could be also viewed like a vector field V along c

$$\begin{aligned} V : I_0 &\longrightarrow TM \\ t &\longrightarrow V(t), \end{aligned} \tag{3.2.5}$$

Where I_0 is an open interval which contains t_0, such that at $t = t_0$; $V(t_0) = V_0$.

Before we reach to the main theorem of this section 2 which is a Fundamental theorem of Riemannian Geometry, let us introduce two key notions of symmetry of a connection ∇ and the compatibility of a connection with respect to the metric on a Riemannian manifold. Let us consider a Riemannian manifold M with its metric $<,>$.

3.2.9 Definition. A connection ∇ on M, is **compatible** to the metric $<,>$. When given a curve c and two parallel vector fields P, P' along the curve c we have:

$$< P, P' >= constante \tag{3.2.6}$$

3.2.10 Proposition. Considering the same Riemannian manifold as in Definition 3.2.9. A connection ∇ on M is compatible with the metric if and only if for any two vectors W_1, W_2 field along the curve c we have:

$$\frac{d}{dt} < W_1, W_2 >=< \frac{DW_1}{dt}, W_2 > + < W_1, \frac{DW_2}{dt} > \quad \forall t \in I. \tag{3.2.7}$$

Proof. Confer chapter 2 in Do Carmo's Book [2] □

3.2.11 Remarks.

1. The compatibility of the connection allows us to differentiate the inner product, by using the product rule.

2. We can also say that a connection ∇ on a Riemannian manifold M^n, is compatible if it preserves the metric, which means that:

$$X < Y, Z >=< \nabla_X Y, Z > + < Y, \nabla_X Z >; \; X, Y, Z \in \mathfrak{X}(M).$$

3.2.12 Definition. A connection ∇ is said to be **symmetric**, when it is torsion free, which means

$$\nabla_X Y - \nabla_Y X = [X, Y] \ \forall X,\ Y \in \mathfrak{X}(M). \tag{3.2.8}$$

3.2.13 Theorem (Levi-Civita connection). *Given a Riemannian manifold M, we can find a unique connection ∇ on M, which satisfies:*

1. *∇ is symmetric.*
2. *∇ is compatible with the Riemannian metric on M.*

3.2.14 Lemma (Koszul formula). Given a connection ∇ that is symmetric and compatible with the metric, we have the following formula:

$$\begin{aligned} 2 < \nabla_X Y, Z > \ &= \ X(< Y, Z >) + Y(< X, Z >) - Z(< X, Y >) - < Y, [X, Z] > - < X, [Y, Z] > \\ &\quad - < Z, [Y, X] > . \end{aligned} \tag{3.2.9}$$

For the prove see appendix A.0.1. Now let us prove Theorem 3.2.13

Proof.

- Assuming the existence of this connection ∇, let us show that it is unique. Since ∇ is symmetric, by lemma it satisfies (3.2.9), by the uniqueness of the inner product, we conclude directly that ∇ is unique.
- Let us prove the existence of this connection which satisfies the two conditions of the theorem. Let us define ∇ like in (3.2.9), and let us show that it verified the 3 conditions of a connection.
  - The first property comes, using the fact that taken $Y_1, Y_2 \in \mathfrak{X}(M)$,we have, $[Y_1 + Y_2, Z] = [Y_1, Z] + [Y_2, Z]$ and also by the linearity of $<,>$ we get the answer.
  - For second one we use the same trick.
  - The property of derivative. Let $f \in \mathcal{D}(M)$, in (3.2.9) replacing Y by fY we have :

    $$\begin{aligned} 2 < \nabla_X (fY), Z > \ &= \ X(< fY, Z >) + fY(< X, Z >) - Z(< X, fY >) - < fY, [X, Z] > \\ &\quad - < X, [fY, Z] > - < Z, [fY, X] > . \end{aligned}$$

    Since

    $$\begin{aligned} X(< fY, Z >) &= X(f < Y, Z >) = X(f) < Y, Z > + fX < Y, Z > \\ Z(< X, fY >) &= Z(f) < X, Y > + fZ < X, Y > \\ [fY, Z] &= fYZ - Z(fY) = fYZ - fZY - Z(f)Y = f\,[Y, Z] - Z(f)Y \\ [fY, X] &= f\,[Y, X] - X(f)Y. \end{aligned}$$

After putting those expressions together, we can write

$$\begin{aligned}
2 < \nabla_X(fY), Z > &= f(X(< Y, Z >) + Y(< X, Z >) - Z(< X, Y >) \\
&\quad - < Y, [X, Z] > - < X, [Y, Z] - < Z, [Y, X] >) \\
&\quad - Z(f) < X, Y > + Z(f) < X, Y > \\
&\quad + X(f) < Y, Z > + X(f) < Y, Z > \\
&= 2f < \nabla_X Y, Z > + 2X(f) < Y, Z > \\
2 < \nabla_X(fY), Z > &= 2 < f\nabla_X Y, Z > + 2 < X(f)Y, Z > \\
2 < \nabla_X(fY), Z > &= 2 < f\nabla_X Y + X(f)Y, Z > \\
2 < \nabla_X(fY) - f\nabla_X Y - X(f)Y, Z > &= 0 \text{ implies } \nabla_X(fY) = f\nabla_X Y + X(f)Y
\end{aligned}$$

Then ∇ acts like a derivative

This ends the proof of Theorem 3.2.13 □

3.2.15 Remark. The connection defined in Lemma 3.2.14, is called the levi Civita-connection on the Riemannian manifold M.

3.2.16 Corollary. For the Levi-Civita connection of M, considered at a point $p \in M$ an open neighborhood $U \subset M$ containing p, and the local coordinates $(x_1, ..., x_n)$, with the basis $(X_1, ..., X_n)$, where $X_i = \frac{\partial}{\partial x_i}(p)$, we have:

1. $\Gamma^m_{ij} = \Gamma^m_{ji}$
2. The relation between the local representation of a Riemannian metric (g_{ij}) and the Christoffel symbols are given by

$$\Gamma^m_{ij} = \frac{1}{2} g^{ml} \left(g_{jl,i} + g_{il,j} - g_{ij,l} \right); \tag{3.2.10}$$

where $\partial_k g_{ij} = \frac{\partial}{\partial x_k}(g_{ij}) = g_{ij,k}$

For the demonstration see appendix A.0.3. Now we are going to introduce one of the fundamental notions in Riemannian geometry, the geodesic which comes from geodesy, the science of measuring the size and shape of the earth. In what follows, we will work with the Levi-Civita connection and thus, for every curve, γ on M, let $\frac{D}{dt}$ be the associated covariant derivative along c, $\nabla_{\gamma'}$.

3.3 Geodesics

3.3.1 Definition. Let $(M^n, < \ , \ >)$ be a Riemannian manifold. A curve, $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ is a **geodesic** if and only if $\frac{D\gamma'}{dt} = \nabla_{\gamma'}\gamma' = 0$; which means γ' is a parallel vector along the curve γ.

If M was embedded in $\mathbb{R}^d$, a geodesic would be curve γ, with zero acceleration vector which means that $\gamma'' = \frac{D\gamma'}{dt} \equiv 0$. A geodesic is said to be **normalized** if $|\gamma'(t)| = 1$.

3.3.2 Geodesic equation. Given a Riemannian manifold $(M^n, < \ , \ >)$, and a geodesic γ, taking the local chart at $p = \gamma(t_0) \in M$, (U, φ), where U is an open that contains p, with its local coordinates $(x^1, ..., x^n)$, let us find the equation satisfied by the geodesic in this local coordinates. the local frame is given by $\left(\frac{\partial}{\partial x_i}\right)$ at the given point p. $\gamma(t) = (x^1(t), ..., x^n(t))$, so
$\gamma'(t) = \frac{d\gamma}{dt} = (\dot{x}^1(t), ..., \dot{x}^n(t)) = \dot{x}^i \frac{\partial}{\partial x_i}$. Since γ is a geodesic, we have $\frac{D\gamma'}{dt} = \nabla_{\gamma'}\gamma' = 0$. Then:

$$\begin{aligned}
\nabla_{\gamma'}\gamma' = 0 \quad &\Longrightarrow \quad \nabla_{\dot{x}^i \frac{\partial}{\partial x_i}} \left(\dot{x}^j \frac{\partial}{\partial x_j} \right) = 0 \\
&\Longrightarrow \quad \dot{x}^i \frac{\partial}{\partial x_i}(\dot{x}^j) \frac{\partial}{\partial x_j} + \dot{x}^i \dot{x}^j \nabla_{\frac{\partial}{\partial x_i}} \left(\frac{\partial}{\partial x_j} \right) = 0 \\
&\Longrightarrow \quad \frac{d}{dt}(\dot{x}^k) \frac{\partial}{\partial x_k} + \dot{x}^i \dot{x}^j \Gamma_{ij}^k \frac{\partial}{\partial x_k} = 0 \\
&\Longrightarrow \quad \left(\frac{d^2 x^k}{dt^2} + \dot{x}^i \dot{x}^j \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k} = 0 \\
\nabla_{\gamma'}\gamma' = 0 \quad &\Longrightarrow \quad \frac{d^2 x^k}{dt^2} + \dot{x}^i \dot{x}^j \Gamma_{ij}^k = 0
\end{aligned}$$

Therefore the equation satisfied by the geodesic in the local frame is given by:

$$\frac{d^2 x^k}{dt^2} + \dot{x}^i \dot{x}^j \Gamma_{ij}^k = 0; \tag{3.3.1}$$

which is a second-order non linear differential equation in x^i.

3.3.3 Examples of geodesics.

- On $\mathbb{R}^n$ geodesics are straights lines, with constant speed.
  this is because if we consider a lines $\gamma(t) = (\gamma_1(t), ..., \gamma_n(t))$ with constant speed it will means that $\gamma'(t) = (\gamma_1'(t), ..., \gamma_n'(t))$, where each components are constants, so $\gamma''(t) \equiv 0$, then γ is a geodesic.

- On $S^n(r)$the sphere of radius r is a Riemannian manifold.
  Since $S^n(r)$ is a Riemannian manifold of $\mathbb{R}^{n+1}$ with the induced metric of $\mathbb{R}^{n+1}$. Let us denote by $\overline{\nabla}$ the Levi-Civita connection of $\mathbb{R}^{n+1}$ and by ∇ the Levi-Civita connection of $S^n(r)$. Let us find the Levi-Civita connection of $S^n(r)$. Using the fact that $S^n(r)$ is embedding in $\mathbb{R}^{n+1}$, let call $h : S^n(r) \longrightarrow \mathbb{R}^{n+1}$ the immersion, we know that $S^n(r)$ is a Riemannian manifold with the metric induced by the euclidean metric on $S^n(r)$, then considering two vectors fields X and Y on $S^n(r)$, which are also vector fields in $\mathbb{R}^{n+1}$, and then $(\nabla_X Y)(p)$ is just the tangent component of $(\overline{\nabla}_X Y)(p)$. Hence we can write: $\nabla_X Y$=Projection $(\overline{\nabla}_X Y)$ on the ambient space. We therefore have:
  $$\overline{\nabla}_X Y = (\overline{\nabla}_X Y)^{\perp} + (\overline{\nabla}_X Y)^{\top}$$
  Since $(\overline{\nabla}_X Y)^{\top} = \nabla_X Y$
  $$\overline{\nabla}_X Y \ = \ (\overline{\nabla}_X Y)^{\perp} + \nabla_X Y \implies \nabla_X Y = \overline{\nabla}_X Y - (\overline{\nabla}_X Y)^{\perp}.$$
  Knowing that the normal vector of $S^n(r)$ at a point p is the simply the line generated by p, let us take the unit normal one $\frac{p}{|p|}$ and hence we have $(\overline{\nabla}_X Y)^{\perp} =< \overline{\nabla}_X Y, \frac{p}{|p|} > \frac{p}{|p|}$. Then the Levi-Civita connection of $S^n(r)$, is given by:
  $$\nabla_X Y = \overline{\nabla}_X Y - < \overline{\nabla}_X Y, \frac{p}{|p|} > \frac{p}{|p|} \tag{3.3.2}$$

Let $\gamma : I \subset \mathbb{R} \longrightarrow S^n(r)$, γ is a geodesic implies that $\nabla_{\gamma'}\gamma' = 0$, using (3.3.2) it means that:

$$\overline{\nabla}_{\gamma'(t)}\gamma'(t) - < \overline{\nabla}_{\gamma'(t)}\gamma'(t), \frac{\gamma(t)}{|\gamma(t)|} > \frac{\gamma(t)}{|\gamma(t)|} = 0 \implies \overline{\nabla}_{\gamma'(t)}\gamma'(t) = < \overline{\nabla}_{\gamma'(t)}\gamma'(t), \frac{\gamma(t)}{|\gamma(t)|} > \frac{\gamma(t)}{|\gamma(t)|}$$
$$\implies \gamma''(t) = \frac{1}{r^2} < \gamma''(t), \gamma(t) > \gamma(t).$$

Hence geodesics on the sphere $S^n(r)$ are curve γ such that:

$$\gamma''(t) = \frac{1}{r^2} < \gamma''(t), \gamma(t) > \gamma(t).$$

- If we consider a particular case of the unit sphere of $\mathbb{R}^3$, geodesics are the great circles. Let us show it.
  Considering the parametrization in polar coordinates as follows:

$$\begin{aligned} h : (0,\pi) \times [0, 2\pi] &\longrightarrow S^2 \\ (\varphi, \theta) &\longrightarrow (sin\varphi cos\theta, sin\varphi sin\theta, cos\varphi). \end{aligned}$$

  Using the following result that we are going to show in Chapter 4, we have the Christophel symbols which are given by: $\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{11} = \Gamma^2_{22} = 0$ and $\Gamma^1_{22} = -\frac{1}{2}sin(2\varphi)$; $\Gamma^2_{12} = \Gamma_2 21 = cot(\varphi)$. Then using (3.3.1), and setting $x^1 = \varphi; x^2 = \theta$ we have the following equation satisfied by the geodesic:

$$\frac{d^2\varphi}{dt^2} + \Gamma^1_{22}\frac{d\theta}{dt}\frac{d\theta}{dt} = 0 \tag{3.3.3}$$
$$\frac{d^2\theta}{dt^2} + 2\Gamma^2_{12}\frac{d\varphi}{dt}\frac{d\theta}{dt} = 0 \tag{3.3.4}$$

  Changing t as a function of θ as follows: $t = a\theta$; $a \neq 0$, so there is not dependence on φ, so φ is viewed like a constant it means that $\frac{d\varphi}{dt} = 0$; $t = a\theta \implies \frac{d\theta}{dt} = \frac{1}{a}$ and then $\frac{d^2\theta}{dt} = 0$. By applying those results on the above equations, (3.3.4) is always true because the right hand side vanishes. (3.3.3) implies that, $0 - \frac{1}{2}sin(2\varphi) \times \frac{1}{a} = 0$, so $sin(2\varphi) = 0$ then $\varphi = \frac{k\pi}{2}$; $k \in \mathbb{Z}$. Since $\varphi \in (0, 2\pi) \implies 0 < k < 2 \implies k = 1$, then $\varphi = \frac{\pi}{2}$. Hence our geodesic is given by

$$\begin{aligned} h_{\frac{\pi}{2}} : [0, 2\pi] &\longrightarrow S^2 \\ \theta &\longrightarrow (sin(\frac{\pi}{2})cos\theta, sin(\frac{\pi}{2})sin\theta, cos(\frac{\pi}{2}) = (cos\theta, sin\theta, 0). \end{aligned}$$

  This is a great circle. If we give another initial conditions such that the initial point does not belong to the equator, we can make some transformation like a rotation to end at the solution. Since on the sphere, it is the only circle which is invariant by such kind of isometry we will reach again to another great circle. Then we can conclude that geodesics are great circles on the sphere.

Having (3.3.1), we could determine geodesics by solving it, which is not an easy task since it is a non linear differential equation. We Know that every second order differential equation can be transformed into a system of first-order differential equations, except to work on M, we will work on TM. setting $\frac{dx^k}{dt} = y^k$, then we have $\frac{dy^k}{dt} = -\dot{x}^i\dot{x}^j\Gamma^k_{ij}$ which lead to the following system of differential equations:

$$\begin{cases} \frac{dx^k}{dt} = y^k \\ \frac{dy^k}{dt} = -\dot{x}^i\dot{x}^j\Gamma^k_{ij} \end{cases} \tag{3.3.5}$$

$k = 1, ..., n$. The solutions of this equation will be seen like a trajectory passing through a point $q \in U$, with velocity $v_q \in T_qU$ in TM. Hence we are interested in the uniqueness and the existence of the solution of (3.3.5) with initial condition, at time t_0 it passes through a point p with a velocity $v_p \in T_pU$. having a curve $\gamma : I \in \mathbb{R} \longrightarrow M$, we can define curve β on TM as follows:

$$\begin{array}{rcl} \beta : I & \longrightarrow & TM \\ t & \longrightarrow & (\gamma(t), \gamma'(t)) \end{array}$$

Hence if γ is a geodesic, then β defined like above should be a geodesic on TM.

3.3.4 Proposition. Let $p \in M$, $v \in T_pM$, there exists a unique geodesic γ_v satisfying:

$$\gamma_v(t_0) = p, \ \dot{\gamma}_v(t_0) = v$$

Proof. Assuming the existence, let us first prove the uniqueness of γ_v. Consider the coordinates (U, φ) on M, the trajectories of γ_v on T_U are defined as follows $t \longrightarrow (\gamma_v(t), \dot{\gamma}_v(t))$, where γ_v is a geodesic. It follows that γ_v satisfies (3.3.5), with initial condition $\gamma_v(t_0) = p, \dot{\gamma}_v(t_0) = v$. Using the Cauchy-Lipschitz theorem, if γ_v exists then it is unique. The existence comes directly if we define a system of equation which verifies (3.3.5) with the initial condition mentioned in Proposition 3.3.4 using the same Cauchy-Lipschitz theorem the existence comes. □

Hence the above Proposition 3.3.4 can be also be rewritten as follows by using theorem 2.2 chapter 3 of Do Carmo [2].

3.3.5 Proposition (Do Carmo [2])**.** Let $(M^n, < \ , \ >)$ be a Riemannian manifold, given $p \in M$, there exist $V \subset M$, V and open that contains p; $\delta > 0, \epsilon_1 > 0$ and $C^{+\infty}$ mapping $\phi : (-\delta, \delta) \times U \longrightarrow M$; $U = \{(q, v) \in T_pM; p \in V; v \in T_qM, |v| < \epsilon_1\}$ such that the curve $t \longrightarrow \gamma(t, q, v), t \in (-\delta, \delta)$ is the unique geodesic of M which at $t = 0$, passes through $q = \gamma(0, q, v)$ with velocity $v = \dot{\gamma}(0, q, v)$ for all q and v element of the tangent space at p, such that $|v| < \epsilon_1$.

3.3.6 Lemma. (Homogeneity of a geodesic) If the geodesic $\gamma(t, q, v)$ is defined on the interval $(-\delta, \delta)$, Then $\gamma(at, q, v)$ is a geodesic defined on $\left(-\frac{\delta}{a}, \frac{\delta}{a}\right)$ and $\gamma(t, q, av) = \gamma(at, q, v)$. Notice that if $a < 0$ we just swap the bounds of the interval.

Proof. Let $h : \left(-\frac{\delta}{a}, \frac{\delta}{a}\right) \longrightarrow h(t) = \gamma(at, q, v)$ be a curve. h is well defined, since $at \in (-\delta, \delta)$. We have $h(0) = \gamma(0, q, v) = q$ and $\frac{dh}{dt}(t) = a\dot{\gamma}(at, q, v)$ implies that $\frac{dh}{dt}(0) = a\dot{\gamma}(0, q, v) = av$. Let us check if h satisfies the property of a geodesic, we have: $\frac{D}{dt}\left(\frac{dh}{dt}(t)\right) = \nabla_{h'(t)}h'(t)$ using the expression of the derivative of h we get $\frac{D}{dt}\left(\frac{dh}{dt}(t)\right) = \nabla_{a\dot{\gamma}(at,q,v)}(a\dot{\gamma}(at, q, v)) = a^2\nabla_{\dot{\gamma}(at,q,v)}(\dot{\gamma}(at, q, v))$. Since $at \in (-\delta, \delta)$ we obtain $\nabla_{\dot{\gamma}(at,q,v)}(\dot{\gamma}(at, q, v)) = 0$, then $\frac{D}{dt}\left(\frac{dh}{dt}(t)\right) = 0$. So h is a geodesic which passed through q with a velocity av, using the uniqueness of geodesic we have $h(t) = \gamma(at, q, v)$, so $\gamma(at, q, v) = \gamma(t, q, av)$. □

Now let us introduce a little bit the notion of exponential map, and some examples of them before ending this section by the properties of minimizing geodesic.

3.3.7 Exponential map. The interest in defining this notion is due to the fact that it helps to parametrize a Riemannian manifold M, locally near any $p \in M$ in terms of a map from the tangent space T_pM to the manifold. This notion of the exponential map is being defined in terms of geodesics.

3.3.8 Definition. Let $P \in M$, and let $U \subset TM$ be an open set given like in Proposition 3.3.5, then the map $exp : U \longrightarrow M$, given by:

$$exp(q, v) = \gamma(1, q, v), \qquad (q, v) \in U;$$

is called the **exponential map** on U.

Using the Lemma 3.3.6, we can see that taken $a = \frac{1}{|v|}$ we can write : $exp(q, v) = \gamma(|v|, q, \frac{v}{|v|})$ in what follows we are going to define the exponential map on an open ball that contains the origin 0 of T_qM, with radius ϵ in T_qM denoted $B_\epsilon(0)$ as follows:

$$exp_q : \ B_\epsilon(0) \longrightarrow M.$$

3.3.9 Proposition. The exponential map is a local diffeomorphism.

Proof. Let us find $d(exp_q)_0$. we have:

$$\begin{aligned} d(exp_q)_0 v &= \frac{d}{dt}(exp_q(tv))|_{t=0} = v; \text{ by the definition of the exponential map;} \\ d(exp_q)_0 &= id. \end{aligned}$$

Hence $d(exp_q)$ has a maximal rank, using the inverse theorem function there exists an open ball in T_qM of radius ϵ with center 0, and an open set $V \subset M$ such that $exp_q : B_\epsilon(0) \longrightarrow V \subset M$ is a diffeomorphism. $\square$

3.3.10 Remarks.

- The map exp is differentiable since γ is differentiable.
- Given a point $q \in M$, exp_q is diffeomorphic from an open subset of TM to an open subset of M.
- The exponential map is not necessarilly defined on all of the tangent space.

3.3.11 Example of exponential map on some manifold.

- The euclidean space $\mathbb{R}^n$
  We know that for a given $p \in \mathbb{R}^n$ we have $T_p\mathbb{R}^n \cong \mathbb{R}^n$, and since geodesic are straight line we have:

$$\begin{aligned} exp_p : \mathbb{R}^n &\longrightarrow \mathbb{R}^n. \\ v &\longrightarrow p + v \end{aligned}$$

- The unit sphere S^2 of $\mathbb{R}^3$
  Since that geodesics are great circles, defining exponential map at a point $(N, v) \in T^2_S$, such that N is the north pole, we will have: $exp_N(2\pi v) = N$. It is easier to view that defining all on T_NS^2 it is not injective because $exp_N(\frac{\pi}{2}v) = exp_N(-\frac{3\pi}{2}v)$. And to make it to be injective we can define it on $B_\pi(0)$ to $S^2 - \{S\}$ the south pole.

3.3.12 Geodesic coordinates. Having the notion of exponential map, it could be useful if we can define a local coordinates which could simplify some computations. A **geodesic coordinates** is local coordinates on Riemannian manifold with a symmetry connection defined by the exponential map on an open ball containing the origin 0 in T_pM and an isomorphism $E : \mathbb{R}^n \longrightarrow T_pM$ given by any orthonormal basis of the tangent space at a fixed point $p \in M$. It means that in this coordinate the Christophel symbols $\Gamma_{ij}^m(p)$ will vanish since p is identified by $(0, 0, .., 0)$, and this will induce the fact that the covariant derivative of a tensor will coincide with the normal derivative.
Hence working on a geodesic coordinates around a point p will induce that (3.3.1) of geodesic in this local frame will be simply reduce to

$$\frac{d^2x^k}{dt^2} = 0$$

which implies that: $x^k = \beta^k t$. It is coming down that there is a correspondence between the straight lines x^k in the domain where the geodesic coordinates are defined an curves $\gamma(t)$. To conclude this having the exponential map we can also define the notion of **normal neighborhood** $V \subset M$, which is the image by the exponential map of an open $U \subset T_pM$, such that the restriction of $exp_{p|_U}$ is a diffeomorphism into V. If U is a open ball on T_pM is hold that $V = exp_p(U)$ is a **normal ball**.

3.3.13 Proposition. Let $p \in M$, we can find a normal neighborhood of p, V, and a constant $\alpha > 0$ such that for every $q \in V$, the exponential map defined on the open ball $B_\alpha(0) \subset T_qM$ is diffeomorphic to an open of M and we have $V \subset exp_q(B_\alpha(0))$

Proof. We use the Proposition 3.3.5 and the Proposition 3.3.9. For more detail see Do.Carmo book's [2] □

3.3.14 Minimizing properties of geodesics. The idea behind this notion is to show that the geodesic which connects two sufficiently close points is the shortest path between them.

3.3.15 Definition. A **piecewise differential curve** β defined on $[a, b]$ is a continuous function, such that it exists a partition of the interval $[a, b]$ into interval $[t_i; t_{i+1}]$, $i \in \{0, ..., n-1\}$ with $t_0 = a, t_n = b$, such that $\beta|_{[t_i;t_{i+1}]}$ are differentiable.

Considering the segment of geodesic $\gamma : [a, b] \longrightarrow M$, denoted by $\mathcal{P}$ the set of piecewise curves joining $\gamma(a)$ and $\gamma(b)$.

3.3.16 Definition. A segment of the geodesic γ defined like above, is called **minimizing** if:

$$l_a^b(\gamma) \leq l_a^b(\beta) \text{ where } \beta \in \mathcal{P}.$$

Hence we can see that geodesics locally minimize the arc length. This hold that every small segment of geodesic is minimal.

3.3.17 Proposition (proposition 3-6 Do.Carmo [2])**.** Let $p \in M$, U a normal neighborhood of p, and $B \subset U$ a normal ball of center p. Let $\gamma : [0, 1] \longrightarrow B$ be a geodesic segment with $\gamma(0) = p$. If $c : [0, 1] \longrightarrow M$ is any piecewise differentiable curve joining $\gamma(0)$ to $\gamma(1)$ then $l_0^1(\gamma) \leq l_0^1(c)$ and if equality holds then $\gamma([0, 1]) = c([0, 1])$.

3.3.18 Proposition. Let M be a Riemannian manifold, and γ a geodesic on M, then γ is regular.

Proof. Let γ be a geodesic, and a point $p \in \gamma$, hence $p \in M$. Defining the exponential map on T_pM, is locally diffeomorphic to and open $V \subset M$ due to Proposition 3.3.9, then $exp_p : U \subset T_pM \longrightarrow V$ is a diffeomorphism, setting V to be a convex neighborhood. Then taking another point q such that $q \in V$ and belong to γ, hence γ can be viewed as a map $[t_0, t_1] \longrightarrow M$ such that $\gamma(t_0) = p, \gamma(t_1) = q$, and $\gamma([t_0, t_1]) \subset V$, $\gamma|_{[t_0,t_1]}$ is the unique geodesic that connects p and q . Consider

$$\begin{aligned} c : [t_0, t_1] &\longrightarrow V \\ t &\longrightarrow exp_p\left(\frac{t-t_0}{t-t_1}v\right). \end{aligned}$$

it comes that c is a differentiable map like composition of two differentiable maps, so c is a curve, we have $c(t_0) = exp_p(0) = p, c(t_1) = exp_p(v) = q$, using Proposition 3.3.17, $l_{t_0}^{t_1}(\gamma) \leq l_{t_0}^{t_1}(c)$. c is regular. By using the definition of the exponential map and Lemma 3.3.6; we find out that c is a geodesic which connect p and q, then $\gamma|_{[t_0,t_1]} \equiv c$ so γ is regular. $\square$

3.3.19 Proposition. If a piecewise differentiable curve $\gamma : [a, b] \longrightarrow M$, with parameter proportional to arc length, has length less or equal to the length of any other piecewise differentiable curve joining $\gamma(a), \gamma(b)$ then γ is a geodesic.

Proof. Let $t \in [a, b]$ and W a normal neighborhood of $\gamma(t)$, since $\gamma(t) \in M$, hence there exists $I_0 \subset [a, b]$, a closed interval such that $\gamma|_{I_0}$ is a piecewise function defined from $I_0 \longrightarrow w$ joining two points of the normal ball. Using the previous proposition, and due to the fact that γ_{I_0} is parametrized by arc length since $l(\gamma|_{I_0})$ is equal to the radial geodesic; we can conclude that $\gamma|_{I_0}$ is a geodesic on I_0, hence it could be extended for all $t \in [a, b]$ $\square$

This proposition holds to that, if a piecewise differentiable curve c is minimizing we can show that c is a geodesic, and this lead to the fact that geodesic are regular.

4. Curvature in Riemannian manifold

The curvature on a Riemannian manifold was introduced by Riemann (see Riemann [8]), the objective was to generalize the notion of Gaussian curvature that we know for surfaces. In what follows we will work with the Riemannian manifold $(M^n, <, >)$, with its Levi-Civita connection.

4.1 Curvature tensor

Before reaching to the definition of curvature tensor, let us make a quick recap on the notion of tensor on a Riemannian manifold.

4.1.1 Definition. A **tensor** T of order r on a Riemannian manifold is a $\mathcal{D}(M)$-multilinear mapping, defined as follows:

$$T : \underbrace{\mathfrak{X}(M) \times \mathfrak{X}(M) \times .... \times \mathfrak{X}(M)}_{r-\text{factors}} \longrightarrow \mathcal{D}(M). \tag{4.1.1}$$

This means that given $Z_1, ..., Z_r$, $T(Z_1, ..., Z_r)$ is a differentiable function over M, and T is linear with respect to each variables.

4.1.2 Definition. The **curvature** denoted by R, is a mapping:

$$\begin{aligned} R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ (X, Y, Z) &\longrightarrow R(X, Y)Z \end{aligned}$$

defined as follows:

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z; \tag{4.1.2}$$

where ∇ is the Riemannian connection of M mentioned in the previous chapter 3 (Levi Civita connection).

This definition can be different from one book to another like in [4], the one we use here is the one defined in [2]. But the idea behind this notion still the same. Let us find the curvature of the Riemannian manifold when $M = \mathbb{R}^n$: $Z \in \mathfrak{X}(\mathbb{R}^n)$, it can be looked like $Z = (z_1, ..., z_n)$ then from (4.1.2) we have :

$$\nabla_Y \nabla_X Z = (Y X z_1, ..., Y X z_n) \tag{4.1.3}$$

$$\nabla_X \nabla_Y Z = (XY z_1, ..., XY z_n) \tag{4.1.4}$$

Hence

$$\begin{aligned} \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z &= (Y X z_1 - XY z_1, ..., Y X z_n - XY z_n) \\ &= ((Y X - XY) z_1, ..., (Y X - XY) z_n) \\ \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z &= -\nabla_{[X,Y]} Z. \end{aligned}$$

Then

$$\begin{aligned} R(X, Y)Z &= -\nabla_{[X,Y]} Z + \nabla_{[X,Y]} Z \\ R(X, Y)Z &= 0 \quad \forall\, X, Y, Z \in \mathfrak{X}(M) \implies R \equiv 0. \end{aligned} \tag{4.1.5}$$

Hence due to (4.1.5), we see the curvature as a tool to measure how much a Riemannian manifold deviate from being euclidean.

4.1.3 Proposition. A curvature defined above has the property that it is a $\mathcal{D}(M)$-tri-linear map. Hence we can see it as follows:

- R is bilinear in $\mathfrak{X}(M) \times \mathfrak{X}(M)$.
- For all $X, Y \in \mathfrak{X}(M)$, $R(X,Y)$ linear from $\mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$.

For the prove See Appendix A.0.2.

4.1.4 Remark. The appearance of $\nabla_{[X,Y]}$ is very relevant to make $R(X,Y)$ linear.

4.1.5 Definition. The **curvature tensor** is the $4-$tensor, defined as follows:

$$(X,Y,Z,T) =< R(X,Y)Z,T > \qquad \forall\ X,Y,Z,T \in \mathfrak{X}(M). \tag{4.1.6}$$

NB It comes that (X,Y,Z,T) is linear with respect to T, and since $R(X,Y)Z$ is tri-linear, we then conclude that (X,Y,Z,T) is a $4-$ tensor. This curvature tensor satisfies some, properties that we are going to see in the following proposition.

4.1.6 Proposition. The curvature tensor (X,Y,Z,T) is:

1. Skew-symmetric with respect to the two first entries and the two last, which means:

$$(X,Y,Z,T) = -(Y,X,Z,T) \tag{4.1.7}$$
$$(X,Y,Z,T) = (Y,X,T,Z). \tag{4.1.8}$$

2. Symmetric with respect to the two first entries and the two last one, which means:

$$(X,Y,Z,T) = (T,Z,X,Y). \tag{4.1.9}$$

For the prove see Appendix A.0.2

4.1.7 Expression of the curvature and curvature tensor in terms of Christophel symbol Γ_{ij}. Let us consider the chart (U,ϕ), with local coordinate $(x^1,...,x^n)$, and $\left(\frac{\partial}{\partial x_i}\right)_{i=1,...,n} = X_i$ the basis. We have:

$$R(X_i,X_j)X_k = \nabla_{X_j}\nabla_{X_i}X_k - \nabla_{X_i}\nabla_{X_j}X_k \quad , \text{ since } \ [X_i,X_j] = 0.$$

Due to the fact that $R(X_i,X_j)X_k \in \mathfrak{X}(M)$, in term of local coordinate we have:

$$R(X_i,X_j)X_k = R^l_{ijk}X_l \tag{4.1.10}$$

Since $\nabla_{X_i}X_k = \Gamma^s_{ik}X_s$, $\nabla_{X_j}X_k = \Gamma^s_{jk}X_s$, we have

$$R(X_i,X_j)X_k = \nabla_{X_j}(\Gamma^s_{ik}X_s) - \nabla_{X_i}(\Gamma^s_{jk}X_s)$$

Since

$$\nabla_{X_j}(\Gamma^s_{ik}X_s) = X_j(\Gamma^s_{ik})X_s + \Gamma^s_{ik}\nabla_{X_j}X_s = X_j(\Gamma^s_{ik})X_s + \Gamma^s_{ik}\Gamma^l_{js}X_l$$

$$\nabla_{X_i}(\Gamma^s_{jk}X_s) = X_i(\Gamma^s_{jk})X_s + \Gamma^s_{jk}\nabla_{X_i}X_s = X_i(\Gamma^s_{jk})X_s + \Gamma^s_{jk}\Gamma^l_{is}X_l$$

Then

$$\begin{aligned} R(X_i, X_j)X_k &= X_j(\Gamma^s_{ik})X_s + \Gamma^s_{ik}\Gamma^l_{js}X_l - X_i(\Gamma^s_{jk})X_s - \Gamma^s_{jk}\Gamma^l_{is}X_l \\ &= X_j(\Gamma^l_{ik})X_l + \Gamma^s_{ik}\Gamma^l_{js}X_l - X_i(\Gamma^l_{jk})X_l - \Gamma^s_{jk}\Gamma^l_{is}X_l \\ R(X_i, X_j)X_k &= \left(X_j(\Gamma^l_{ik}) + \Gamma^s_{ik}\Gamma^l_{js} - X_i(\Gamma^l_{jk}) - \Gamma^s_{jk}\Gamma^l_{is}\right) X_l \end{aligned} \tag{4.1.11}$$

Hence using (4.1.10) and (4.1.11) we get the components of curvature in $\left(\frac{\partial}{\partial x_i}\right)$, which are given by:

$$R^l_{ijk} = \Gamma^s_{ik}\Gamma^l_{js} - \Gamma^s_{jk}\Gamma^l_{is} + \frac{\partial}{\partial x_j}(\Gamma^l_{ik}) - \frac{\partial}{\partial x_i}(\Gamma^l_{jk}) \tag{4.1.12}$$

Using (4.1.12), let us write the expression of the curvature tensor, we have:

$$\begin{aligned} < R(X_i, X_j)X_k, X_t > &= < R^l_{ijk}X_l, X_t > \\ &= R^l_{ijk} < X_l, X_t >= R^l_{ijk}g_{lt} \end{aligned}$$

Hence the components of the curvature tensor in $\left(\frac{\partial}{\partial x_i}\right)$ are R_{ijkl}, define by:

$$R_{ijkt} = R^l_{ijk}g_{lt} \tag{4.1.13}$$

4.1.8 Remark.

1. The Proposition 4.1.6 in local coordinate can be written , as follow : -
   - Skew-symmetric or antisymmetric : $R_{ijkt} = -R_{jikt} = R_{jitk}$
   - Symmetric : $R_{ijkt} = R_{tkji}$
2. Due to the antisymmetric, we can see that if the two first entries or the two last entries are identical, the corresponding components of the curvature tensor will vanish.ie

$$\text{if } i = j \text{ or } k = t \implies R_{iikt} = 0 \text{ or } R_{ijkk} = 0$$

3. this Proposition 4.1.6 is very useful when we want to calculate the components of the bending tensor because it allows us to reduce the number of operations to perform and to deduce the other components using the antisymmetric and symmetric properties of the bending tensor.

4.1.9 Examples.

1. The curvature in euclidean geometry is equal to zero, since $R \equiv 0$
2. The unit sphere S^2 of $\mathbb{R}^3$, let us compute the components of the curvature tensor

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$$

Let us consider the parametrization in polar coordinates:

$$\begin{aligned} h : (0, \pi) \times [0, 2\pi] &\longrightarrow S^2 \\ (\varphi, \theta) &\longrightarrow (sin\varphi cos\theta, sin\varphi sin\theta, cos\varphi) \end{aligned}$$

For a given point p, we have T_pS^2=span$\{\frac{\partial}{\partial\varphi}(p), \frac{\partial}{\partial\theta}(p)\}$, and then noted by $g = \begin{pmatrix} g_{\varphi\varphi} & g_{\varphi\theta} \\ g_{\varphi\theta} & g_{\theta\theta} \end{pmatrix}$ the matrix associate to the metric at p, then we have $g_{\varphi\varphi} =< \frac{\partial}{\partial\varphi}(p), \frac{\partial}{\partial\varphi}(p) >,\ g_{\theta\theta} =< \frac{\partial}{\partial\theta}(p), \frac{\partial}{\partial\theta}(p) >$ $, g_{\varphi\theta} = g_{\theta\varphi} =< \frac{\partial}{\partial\varphi}(p), \frac{\partial}{\partial\theta}(p) >$. We have:

$$\begin{aligned} \frac{\partial}{\partial\varphi}(p) &= (cos\varphi cos\theta, cos\varphi sin\theta, -sin\varphi) \\ \frac{\partial}{\partial\theta}(p) &= (-sin\varphi sin\theta, sin\varphi cos\theta, 0) \end{aligned}$$

Hence $g_{\varphi\varphi} = 1,\ g_{\theta\theta} = sin^2\varphi,\ g_{\varphi\theta} = 0$, so $g = \begin{pmatrix} 1 & 0 \\ 0 & sin^2\varphi \end{pmatrix}$ and then $g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{sin^2\varphi} \end{pmatrix}$.

. In what follow let us identified $\frac{\partial}{\partial\varphi}(p) = \frac{\partial}{\partial x_1}(p), \frac{\partial}{\partial\theta}(p) = \frac{\partial}{\partial x_2}(p), g_{\varphi\varphi} = g_{11}, g_{\theta\theta} = g_{22}, g_{\varphi\theta} = g_{12}$.

Using (3.2.10) of chapter 2 section 2, let us find the christophel symbols $\Gamma^1_{11}, \Gamma^1_{22}, \Gamma^1_{12}, \Gamma^2_{12}, \Gamma^2_{11}, \Gamma^2_{22}$ associate to the connection of S^2:

$$\begin{aligned} \Gamma^1_{11} &= \frac{1}{2}g^{1l}\left(g_{1l,1} + g_{1l,1} - g_{11,l}\right) = 0 = \Gamma^1_{12} = \Gamma^1_{21} \text{ Since } g_{11} = 1,\ g^{12} = 0,\ g_{21} = 0 \\ \Gamma^2_{11} &= \frac{1}{2}g^{2l}\left(g_{1l,1} + g_{1l,1} - g_{11,l}\right) = 0 \text{ Since } g_{11} = 1,\ g^{21} = 0 \\ \Gamma^2_{22} &= \frac{1}{2}g^{2l}\left(g_{2l,2} + g_{2l,2} - g_{22,l}\right) = 0 \text{ Since } g_{22} \text{ depend on } \varphi,\ g^{12} = 0 \\ \Gamma^1_{22} &= \frac{1}{2}g^{1l}\left(g_{2l,2} + g_{2l,2} - g_{22,l}\right) = -\frac{1}{2}g_{22,1} = -\frac{1}{2}\frac{\partial}{\partial\varphi}\left(sin^2\varphi\right) = -\frac{1}{2}sin(2\varphi) \\ \Gamma^2_{12} &= \frac{1}{2}g^{2l}\left(g_{2l,1} + g_{1l,2} - g_{12,l}\right) = \frac{1}{2}g^{22}g_{22,1} = \frac{1}{2sin^2\varphi}2sin\varphi cos\varphi = cot\varphi = \Gamma^2_{21} \end{aligned}$$

Components of the curvature tensor $R_{ijkt} = R^l_{ijk}g_{lt}$. They are 16 terms to find, due to remark4.1.8 we just need to find 8 of them. Using the same remark they are 7 terms many which vanish, except R_{2121}

$$R_{2121} = R^1_{212}g_{11} + R^2_{212}g_{21} = R^1_{212}g_{11} \text{ since } g_{21} = 0$$

Using (4.1.12), let us find R^1_{212}

$$\begin{aligned} R^1_{212} &= \Gamma^s_{22}\Gamma^1_{1s} - \Gamma^s_{12}\Gamma^1_{2s} + \frac{\partial}{\partial x_1}(\Gamma^1_{22}) - \frac{\partial}{\partial x_2}(\Gamma^1_{12}) \\ &= \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{21} - \Gamma^2_{12}\Gamma^1_{22} + \frac{\partial}{\partial x_1}(\Gamma^1_{22}) \end{aligned}$$

Since $\Gamma^1_{11} = \Gamma^1_{12} = 0$

$$\begin{aligned} R^1_{212} &= -\Gamma^2_{12}\Gamma^1_{22} + \frac{\partial}{\partial x_1}(\Gamma^1_{22}) \\ &= -\left(-\frac{sin2\varphi}{2}\right)\frac{cos\varphi}{sin\varphi} + \frac{\partial}{\partial x_1}\left(-\frac{sin2\varphi}{2}\right) \\ &= cos^2\varphi - cos2\varphi = cos^2\varphi - cos^2\varphi + sin^2\varphi \\ R^1_{212} &= sin^2\varphi \end{aligned}$$

So $R_{2121} = sin^2\varphi$

Hence the components of the curvature tensor in local frame $\frac{\partial}{\partial\varphi}(p), \frac{\partial}{\partial\theta}(p)$ are :

$$R_{2121} = sin^2\varphi, R_{1221} = -sin^2\varphi, R_{2112} = -sin^2\varphi, R_{1212} = sin^2\varphi$$

and all others terms vanish.

4.1.10 Proposition. The curvature and the curvature tensor are invariant under local isometries. Which means that given two Riemannian manifolds M, M' and $\phi: \ M \longrightarrow M'$, a local isometries, we have:

$$(\phi_* X, \phi_* Y, \phi_* Z, \phi_* T) = (X, Y, Z, T) \ \ \forall X, Y, Z, T \in \mathfrak{X}(M) \tag{4.1.14}$$

where ϕ_* denoted the differential map between $\mathfrak{X}(M)$ and $\mathfrak{X}(M')$

4.1.11 Lemma. Let M and M' be two Riemannian manifolds and $\phi: \ M \longrightarrow M'$ an isometric, and ∇ a Riemannian connection on M' such that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{X}(M) \times \mathfrak{X}(M) \arrow[r, dotted, "\nabla"] \arrow[d, "\phi_* \times \phi_*"'] & \mathfrak{X}(M) \arrow[d, "\phi_*"] \\
\mathfrak{X}(M') \times \mathfrak{X}(M') \arrow[r, "\nabla"'] & \mathfrak{X}(M')
\end{tikzcd}$$

then ∇ is the Riemannian connection on M, and

$$\phi_* \nabla_X Y = \nabla_{\phi_* X} \phi_* Y \quad \forall X, Y \in \mathfrak{X}(M) \tag{4.1.15}$$

(4.1.15) come to the fact that the above diagram is commutative, because: $\phi_* \nabla = \nabla \phi_* \phi_*$, and then taken $X, Y \in \mathfrak{X}(M)$ we have:

- For the right side of the equality: $\phi_* \nabla(X, Y) = \phi_* \nabla_X Y$.
- For left side we have: $\nabla \phi_* \phi_*(X, Y) = \nabla(\phi_* X, \phi_* Y) = \nabla_{\phi_* X} \phi_* Y$.

Hence equal the right path and the left path we have (4.1.15). Let now prove proposal 4.1.10.

Proof. We have:

$$(\phi_* X, \phi_* Y, \phi_* Z, \phi_* T) \quad = \quad < R(\phi_* X, \phi_* Y)\phi_* Z, \phi_* T >$$

Let us evaluate the quantity $R(\phi_* X, \phi_* Y)\phi_* Z$:

$$R(\phi_* X, \phi_* Y)\phi_* Z \quad = \quad \nabla_{\phi_* Y} \nabla_{\phi_* X}(\phi_* Z) - \nabla_{\phi_* X} \nabla_{\phi_* Y}(\phi_* Z) + \nabla_{[\phi_* X, \phi_* Y]} \phi_* Z$$

Then using (4.1.15) twice on the expressions $\nabla_{\phi_* Y} \nabla_{\phi_* X}(\phi_* Z)$, $\nabla_{\phi_* X} \nabla_{\phi_* Y}(\phi_* Z)$ we have :

$$\nabla_{\phi_* Y} \nabla_{\phi_* X}(\phi_* Z) = \nabla_{\phi_* Y}(\phi_* \nabla_X Z) = \phi_* \nabla_Y \nabla_X Z$$

$$\nabla_{\phi_* X} \nabla_{\phi_* Y}(\phi_* Z) = \nabla_{\phi_* X}(\phi_* \nabla_Y Z) = \phi_* \nabla_X \nabla_Y Z$$

So

$$\begin{aligned} R(\phi_*X,\phi_*Y)\phi_*Z &= \phi_*\nabla_Y\nabla_X Z-\phi_*\nabla_X\nabla_Y Z+\nabla_{[\phi_*X,\phi_*Y]}\phi_*Z \\ &= \phi_*\nabla_Y\nabla_X Z-\phi_*\nabla_X\nabla_Y Z+\nabla_{\phi_*[X,Y]}\phi_*Z \\ &= \phi_*\nabla_Y\nabla_X Z-\phi_*\nabla_X\nabla_Y Z+\phi_*\nabla_{[X,Y]}Z \\ R(\phi_*X,\phi_*Y)\phi_*Z &= \phi_*R(X,Y)Z \end{aligned} \tag{4.1.16}$$

Hence using (4.1.16), we get

$$\begin{aligned} (\phi_*X,\phi_*Y,\phi_*Z,\phi_*T) &= <\phi_*R(X,Y)Z,\phi_*T> \\ &= <R(X,Y)Z,T> \quad \text{, since } \phi \text{ is an isometric} \\ (\phi_*X,\phi_*Y,\phi_*Z,\phi_*T) &= (X,Y,Z,T) \end{aligned}$$

□

4.2 Sectional curvature

This notion is likely correlated to the notion of curvature tensor because, given the sectional curvature for all $\sigma \subset T_pM$, (σ a subspace of dimension2), we can define curvature tensor as we are going to see. Let us consider a vector space $\mathcal{V}$, given $x,y \in \mathcal{V}$, the area of the parallelogram generated by x,y, can be express as follow:

$$|x\wedge y| = \sqrt{|x|^2.|y|^2-<x,y>^2}. \tag{4.2.1}$$

In fact (4.2.1) come directly from the definition.

4.2.1 Definition. Given a point $p\in M$, a two dimensional subspace $\sigma\subset T_pM$, generated by x,y two linearly independent tangent vectors in T_pM. The **sectional curvature** of σ at p; is a real number denoted $K(p,\sigma)$ defined as follow:

$$K(p,\sigma)=K(x,y)=\frac{(x,y,x,y)}{|x\wedge y|^2}. \tag{4.2.2}$$

4.2.2 Proposition. $K(p,\sigma)=K(x,y)$ does not depend on the choice of the basis which generated σ.

Proof. Let $\{x',y'\}$ another basis of σ, so it exist a transition matrix $P=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$(inversible matrix) between the basis $\{x,y\}$ and $\{x',y'\}$ in such a way that $\begin{cases} x'=ax+by \\ y'=cx+dy\end{cases}$; let us show : $K(x',y')=\ K(x,y)$. We have

$$K(x',y') = \frac{(ax+by,cx+dy,ax+by,cx+dy)}{|(ax+by)\wedge(cx+dy)|^2}$$

Let's decompose the denominator,

$$\begin{aligned} |(ax+by)\wedge(cx+dy)|^2 &= |ax\wedge(cx+dy)+by\wedge(cx+dy)|^2 \\ &= |ad(x\wedge y)+bc(y\wedge x)|^2 \ ; \text{ since } ax\wedge cx = 0 = by\wedge dy \\ &= |ad(x\wedge y)-bc(x\wedge y)|^2 \\ |(ax+by)\wedge(cx+dy)| &= (ad-bc)^2|x\wedge y|^2. \end{aligned} \tag{4.2.3}$$

Let's decompose the numerator, by using the fact that the curvature tensor is a $4-$ tensor, which as some properties (see Proposition 4.1.6)

$$\begin{aligned}
(ax+by,cx+dy,ax+by,cx+dy) &= (ax,cx+dy,ax+by,cx+dy)+(by,cx+dy,ax+by,cx+dy)\\
&= (ax,dy,ax+by,cx+dy)+(by,cx,ax+by,cx+dy)\\
&= (ax,dy,ax,cx+dy)+(ax,dy,by,cx+dy)+(by,cx,ax,cx+dy)\\
&\quad +(by,cx,by,cx+dy)\\
&= (ax,dy,ax,dy)+(ax,dy,by,cx)+(by,cx,ax,dy)+(by,cx,by,cx)\\
&= (ax,dy,ax,dy)-(ax,dy,cx,by)-(cx,by,ax,dy)+(cx,by,cx,by)\\
&= (ad)^2(x,y,x,y)-2adcb(x,y,x,y)+(cb)^2(x,y,x,y)\\
&= ((ad)^2-2adcb+(cb)^2)(x,y,x,y)\\
(ax+by,cx+dy,ax+by,cx+dy) &= (ad-cb)^2(x,y,x,y). \qquad (4.2.4)
\end{aligned}$$

Hence using (4.2.3) and (4.2.4), we have:

$$K(x',y')=\frac{(ad-cb)^2(x,y,x,y)}{(ad-bc)^2|x\wedge y|^2}=\frac{(x,y,x,y)}{|x\wedge y|^2}=K(x,y).$$

$\square$

4.2.3 Remark.

1. The sectional curvature at a point p is not a $2-$ tensor, because we have:

$$K(p,\sigma)=K(2x,y)=K(x,y)\neq 2K(x,y)$$

2. Definition 4.2.1, allows us to define at a point $p\in M$ the sectional curvature of a plane Π spanX,Y in T_pM as the Gaussian curvature, that we are going to define now.

Consider $p\in M$, and $X,Y\in T_pM$, since $T_pM\cong\mathbb{R}^n$, an knowing that T_pM =Span$\{\frac{\partial}{\partial x_i}\}$, we have $X=x^i\frac{\partial}{\partial x_i}, Y=y^j\frac{\partial}{\partial x_j}$, and then we get:

$$\begin{aligned}
K(X,Y) &= \frac{(X,Y,X,Y)}{|X\wedge Y|^2}\\
&= \frac{(X,Y,X,Y)}{<X,X>.<Y,Y>-<X,Y>^2}\\
&= \frac{\left(x^i\frac{\partial}{\partial x_i},y^j\frac{\partial}{\partial x_j},x^k\frac{\partial}{\partial x_k},y^l\frac{\partial}{\partial x_l}\right)}{<x^i\frac{\partial}{\partial x_i},x^k\frac{\partial}{\partial x_k}>.<y^j\frac{\partial}{\partial x_j},y^l\frac{\partial}{\partial x_l}>-<x^i\frac{\partial}{\partial x_i},y^j\frac{\partial}{\partial x_j}>.<x^k\frac{\partial}{\partial x_k},y^l\frac{\partial}{\partial x_l}>}\\
&= \frac{R_{ijkl}x^iy^jx^ky^l}{x^ix^kg_{ik}y^jy^lg_{jl}-x^iy^jx^ky^lg_{ij}g_{kl}}\\
K(X,Y) &= \frac{R_{ijkl}x^iy^jx^ky^l}{(g_{ik}g_{jl}-g_{ij}g_{kl})x^iy^jx^ky^l}. \qquad (4.2.5)
\end{aligned}$$

Then (4.2.5) give us the sectional curvature of Π.

4.2.4 Remarks.

1. From (4.2.5), we can see that the sectional curvature of a plan generated by two independent linear vectors is a ratio of two tensors.

2. Directly for (4.2.5), if we multiply the metric by a constant α, the Christophel symbol will remain the same because it also involves the inverse of the matrix associate to the metric, hence by the formula 4.1.12 we see that the components of the curvature will not change, and due to formula 4.1.13, we can see that the components of the curvature will be multiply be this α, since the denominator of (4.2.5) will be scale by α^2, then the sectional curvature of the plane generated by X, Y will be scale by α^{-1}.

3. Knowing that the denominator is the area of the parallelogram spanned by X, Y, which is twice the area of the triangle with vertex $0, X, Y$. The sectional curvature expresses the deviation of the area of a triangle in the manifold from that in the tangent space.

4. The sectional curvature $K(X, Y)$ can then be shown to be the Gauss curvature of the surface Π generated by X, Y with the induced metric.

4.2.5 Example. Computation of the sectional curvature

- Sphere S^2

  Given a point $p \in S^2$, $T_pS^2 \cong \mathbb{R}^2$, hence taken $\sigma \subset T_pS^2$, spanned by X, Y, element of T_pS^2. So σ is a subspace of 2-dimension include in a 2-dimension vector space, then $\sigma = T_pM$, which is generated by e_1, e_2 canonical basis of $\mathbb{R}^2$. Then:

$$\begin{aligned} K(p, T_pS^2) = K(e_1, e_2) &= \frac{R_{1212}}{g_{11}g_{22} - g_{12}g_{12}} \\ &= \frac{sin^2\varphi}{sin^2\varphi} \\ K(p, T_pM) &= 1 \end{aligned}$$

- The upper half-plane

$$\mathbb{R}^2_+ = \{(x, y)\mathbb{R}^2|; y > 0\}$$

  The metric here is the metric of Lobatchevski's non-euclidean geometry, given by

$$g_{11} = g_{22} = \frac{1}{y^2}, g_{12} = g_{21}$$

  After performing operations we obtain the Christophel symbol of the connection is given by

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^1_{22} = 0, \Gamma^2_{11} = \frac{1}{y},\ \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\frac{1}{y}$$

  The components of the curvature tensor all vanish, except $R_{1212} = R_{2121} = -R_{2112} = -R_{1221}$.

We have:

$$
\begin{aligned}
R_{2121} &= R^1_{212}g_{11} + R^2_{212}g_{21} \\
&= R^1_{212}g_{11} \quad \text{since } g_{21} = 0 \\
&= \Gamma^s_{22}\Gamma^1_{1s} - \Gamma^s_{12}\Gamma^1_{2s} + \frac{\partial}{\partial x}(\Gamma^1_{22}) - \frac{\partial}{\partial y}(\Gamma^1_{12}) \\
&= \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{21} - \Gamma^2_{12}\Gamma^1_{22} - \frac{\partial}{\partial y}(\Gamma^1_{12}) \\
&= \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{21} - \frac{\partial}{\partial y}(\Gamma^1_{12}) \\
R_{2121} &= -\frac{1}{y^4}.
\end{aligned}
$$

At a point $p \in \mathbb{R}^2_+$, $T_p\mathbb{R}^2_+ = \mathbb{R}^2$, then:

$$
\begin{aligned}
K(p, T_p\mathbb{R}^2_+) &= \frac{R_{1212}}{g_{11}g_{22} - g_{12}g_{12}} \\
&= \frac{-\frac{1}{y^4}}{\frac{1}{y^4}} \\
K(p, T_p\mathbb{R}^2_+) &= -1
\end{aligned}
$$

From Example 4.2.5 , we can see that both of those Riemannian manifolds have their sectional curvatures are constants, but it is not all Riemannian manifold with their Levi-Civita whose as this property, now we are looking for properties which could be useful to know such types of Riemannian manifolds.

4.2.6 Proposition. Consider M a Riemanman manifold and p a point of M, and a tri-linear mapping $R' : \ T_pM \times T_pM \times T_pM \longrightarrow T_pM$, such that

$$(R'(X, Y, W), Z) = (X, Y, W, Z)' =< X, W >< Y, Z > - < Y, W >< X, Z > \tag{4.2.6}$$

for all $X, Y, W, Z \in T_pM$. Then M has constant sectional curvature equal to K_0 if and only if $R = K_0R'$, where R is the curvature of M.

First of all by the definition of R', If in (4.2.6):

- We change the role of X and Y we get the antisymmetric of $4-$tensor $(\ , \ , \ , \ )'$ respect to the two first entries

$$(Y, X, W, Z) =< Y, W >< X, Z > - < X, W >< Y, Z >= -(X, Y, W, Z)'$$

- We change the role of W and Z, we get the antisymmetric of $4-$tensor $(\ , \ , \ , \ )'$ respect to the two last entries.

$$(X, Y, Z, W)' =< X, Z >< Y, W > - < Y, Z >< X, W >= (X, Y, W, Z)'$$

- We use the symmetric of the inner product we have the symmetric of $4-$tensor $(\ , \ , \ , \ )'$.

Hence $(\ , \ , \ , \ )'$ satisfies proposition A.0.5, so it a curvature tensor on M.

Proof. We have : $(X, Y, X, Y)' =< X, X >< Y, Y > - < X, Y >^2$, suppose M has a constant sectional curvature equal to K_0, let $p \in M$, and $\sigma \in T_pM$ generated by X, Y, then

$$K(p, \sigma) = K_0 = \frac{< R(X, Y)X, Y >}{|X|^2|Y|^2 - < X, Y >^2} \implies < R(X, Y)X, Y >= K_0(X, Y, X, Y)'$$

Since knowing the sectional curvature for all $\sigma \in T_pM$, permit us to define the curvature tensor, then for all X, Y, W, Z we have

$$(R(X, Y)W, Z) = K_0(R'(X, Y, W), Z) \implies R(X, Y)W = K_0R'(X, Y, W)$$

then $R = K_0R'$. the converse it obvious it come directly from the calculation. □

To make a fee summary about sectional curvature, we can see that this notion given at a point p doesn't depends on the basic which generated the plane include in the tangent space at p, Hence if we change the plane we also change the sectional curvature. Knowing that if $dim(T_pM) = n$ there are $\frac{n(n-1)}{2}$ independent planes of dimension two, so $\frac{n(n-1)}{2}$ sectional curvatures to compute at point p. Hence some of them appear with such frequencies that they deserve special names, which reach us to the notion of Ricci curvature and scalar curvature.

4.3 Ricci curvature and scalar curvature.

4.3.1 Definition. Let $X = Z_n$ be a unit vector in T_pM; we take an orthonormal basis $Z_1, Z_2, ..., Z_n - i$ of the hyperplane in T_pM orthogonal to X, the **Ricci curvature** in the direction X at p denoted $Ric_p(X)$, is the mean of the tensor curvature $(X, Z_i, X, Z_i)_{i \in 1,...,n-1}$, it is given by:

$$Ric_p(X) = \frac{1}{n-1} \sum_{i=1}^{n-1} (X, Z_i, X, Z_i) \tag{4.3.1}$$

4.3.2 Definition. The **scalar curvature** at p denoted by $k(p)$ is the mean of the Ricci curvature at p in the directions ${Z_j}_{j=1,...,n}$ at p :

$$k(p) = \frac{1}{n} \sum_{j}^{n} Ric_p(Z_j) \tag{4.3.2}$$

The definition 4.3.1 can be also write in function of sectional curvature,

$$Ric_p(X) = \frac{1}{n-1} \sum_{i=1}^{n-1} K_p(X, Z_i).|X \wedge Z_i|$$

then it is invariant in the choose of orthogonal basis.

5. Conclusion

Our main purpose in this work was to provide some basic concepts in Riemannian geometry. For that we have defined first some basic notions on differential geometry, followed by the definition of Riemannian manifold. we have proved the existence of a unique connection on Riemannian manifold: the Levi-Civita connection. Then we have introduced one of the fundamental notions of the Riemannian geometry: geodesics. we have proved that geodesics are solutions of an ODE of second-order, and that the solution is unique if we have specified the initial condition. We also showed some properties of minimizing geodesic and then proved the regularity of geodesics. We ended introducing the second fundamental notion: curvature. We proved that defining the sectional curvature on all subspace of dimensional two, it can be sufficient to define the curvature tensor. Therefore, we moved to the properties of minimizing geodesic that we can use to define a distance on the Riemannian manifold. We conclude studying the topology induced by the distance and introducing the Hopf-Rinow Theorem.

Appendix A. Some additional data

A.0.1 Proof of Koszul formula. Lemma 3.2.14

Proof.

$$\begin{aligned} X(<Y,Z>) &= <\nabla_X Y, Z> + <Y, \nabla_X Z> && \text{(A.0.1)}\\ Y(<X,Z>) &= <\nabla_Y X, Z> + <X, \nabla_Y Z> && \text{(A.0.2)}\\ Z(<X,Y>) &= <\nabla_Z X, Y> + <X, \nabla_Z Y> && \text{(A.0.3)} \end{aligned}$$

Hence adding (A.0.1) with (A.0.2) and subtract it to (A.0.3), we have:

$$\begin{aligned} X(<Y,Z>) + Y(<X,Z>) - Z(<X,Y>) &= <\nabla_X Y, Z> + <Y, \nabla_X Z> + <\nabla_Y X, Z> \\ &\quad - <Z, \nabla_X Y> + <Z, \nabla_X Y> + <X, \nabla_Y Z> \\ &\quad - <\nabla_Z X, Y> - <X, \nabla_Z Y> \\ &= <\nabla_X Y, Z> + <Y, \nabla_X Z> - <\nabla_Z X, Y> \\ &\quad + <X, \nabla_Y Z> - <X, \nabla_Z Y> + <\nabla_Y X, Z> \\ &\quad - <Z, \nabla_X Y> \\ \text{Due to the symmetry of } <,>; & \\ X(<Y,Z>) + Y(<X,Z>) - Z(<X,Y>) &= 2<\nabla_X Y, Z> + <Y, \nabla_X Z> - <Y, \nabla_Z X> \\ &\quad + <X, \nabla_Y Z> - <X, \nabla_Z Y> + <Z, \nabla_Y X> \\ &\quad - <Z, \nabla_X Y> \\ X(<Y,Z>) + Y(<X,Z>) - Z(<X,Y>) &= 2<\nabla_X Y, Z> + <Y, \nabla_X Z - \nabla_Z X> \\ &\quad + <X, \nabla_Y Z - \nabla_Z Y> + <Z, \nabla_Y X - \nabla_X Y> \end{aligned}$$

Since ∇ is symmetric; $\nabla_X Z - \nabla_Z X = [X,Z]\,, \nabla_Y Z - \nabla_Z Y = [Y,Z]\,, \nabla_Y X - \nabla_X Y = [Y,X]$

$$\begin{aligned} X(<Y,Z>) + Y(<X,Z>) - Z(<X,Y>) &= 2<\nabla_X Y, Z> + <Y, [X,Z]> + <X, [Y,Z]> \\ &\quad + <Z, [Y,X]> \end{aligned}$$

Then

$$2<\nabla_X Y, Z> = X(<Y,Z>) + Y(<X,Z>) - Z(<X,Y>) - <Y,[X,Z]> - <X,[Y,Z]> - <Z,[Y,X]>$$

This ends the proof of the lemma. □

A.0.2 Proof of Proposition 4.1.6 .

Proof. 1. Skew-symmetric respect to the two first entries and the two last entries.

- Skew-symmetric respect to the two first entries is obvious, since $R(X,Y)Z = -R(Y,X)Z$ then $(X,Y,Z,T) =< R(X,Y)Z, T >=< -R(Y,X)Z, T >= -(Y,X,Z,T)$, which shows (4.1.7).

- Skew-symmetric respect to the two last entries

$$\begin{aligned}(X,Y,Z,T)=(Y,X,T,Z) &\iff (X,Y,Z,T)=-(X,Y,T,Z)\\ &\quad\ \text{Taken } T=Z\\ &\iff (X,Y,Z,Z)=-(X,Y,Z,Z)\\ &\iff (X,Y,Z,Z)=0\end{aligned}$$

Hence (4.1.8) $\iff (X,Y,Z,Z)=0$. We have:

$$\begin{aligned}(X,Y,Z,Z) &= <R(X,Y)Z,Z>\\ &= <\nabla_Y\nabla_X Z,Z> - <\nabla_X\nabla_Y Z,Z> + <\nabla_{[X,Y]}Z,Z>\end{aligned}$$

using the fact that the connection is compatible, we have

$$\begin{aligned}<\nabla_Y\nabla_X Z,Z> &= Y<\nabla_X Z,Z> - <\nabla_X,\nabla_Y Z>\\ -<\nabla_X\nabla_Y Z,Z> &= -X<\nabla_Y Z,Z> + <\nabla_Y Z,\nabla_X Z>\\ <\nabla_{[X,Y]}Z,Z> &= [X,Y]<Z,Z> - <\nabla_{[X,Y]}Z,Z> \ \text{this implies}\\ <\nabla_{[X,Y]}Z,Z> &= \frac{1}{2}[X,Y]<Z,Z>\end{aligned}$$

Summing those expressions implies

$$(X,Y,Z,Z)=Y<\nabla_X Z,Z> -X<\nabla_Y Z,Z> +\frac{1}{2}[X,Y]<Z,Z> \tag{A.0.4}$$

Since

$$\begin{aligned}<\nabla_X Z,Z> &= X<Z,Z> - <Z,\nabla_X Z> \Longrightarrow <\nabla_X Z,Z>=\frac{1}{2}X<Z,Z>\\ <\nabla_Y Z,Z> &= \frac{1}{2}Y<Z,Z>\end{aligned}$$

Replacing $<\nabla_X Z,Z>, <\nabla_Y Z,Z>$ by their above expressions in equationA.0.4, we have:

$$\begin{aligned}(X,Y,Z,Z) &= \frac{1}{2}((YX-XY)<Z,Z>)+\frac{1}{2}[X,Y]<Z,Z>\\ &= -\frac{1}{2}[X,Y]<Z,Z> +\frac{1}{2}[X,Y]<Z,Z>\\ (X,Y,Z,Z) &= 0.\end{aligned}$$

Which ends the anti-symmetric respect to the last two entries.

2. Symmetric

   Bianchi identity: $R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0$. Using this identity we have directly $<R(X,Y)Z+R(Y,Z)X+R(Z,X)Y,T>=0.$, and this implies $(X,Y,Z,T)+(Y,Z,X,T)+(Z,X,Y,T)=0$. Then using this final expression by exchanging the position of the vector fields X,Y,Z,T 4 times, and make the summation of the 4 equations that we will have, we will get (4.1.9)

□

Prove of Proposition 4.1.3

- Bi-linearity in $\mathfrak{X}(M) \times \mathfrak{X}(M)$.
  Let $X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(M)$, and $f \in \mathcal{D}(M)$. We need to show that:

$$R(fX_1 + X_2, Y_1) = fR(X_1, Y_1) + R(X_2, Y_1) \tag{A.0.5}$$
$$R(X_1, fY_1 + Y_2) = fR(X_1, Y_1) + R(X_2, Y_1). \tag{A.0.6}$$

We have:

$$\begin{aligned} R(fX_1 + X_2, Y_1) &= \nabla_{Y_1} \nabla_{fX_1 + X_2} - \nabla_{fX_1 + X_2} \nabla_{Y_1} + \nabla_{[fX_1 + X_2, Y_1]} \\ &= \nabla_{Y_1} \left(f \nabla_{X_1} + \nabla_{X_2} \right) - \left(f \nabla_{X_1} + \nabla_{X_2} \right) \nabla_{Y_1} + \nabla_{[fX_1 + X_2, Y_1]}. \end{aligned}$$

Since $[fX_1 + X_2, Y_1] = f[X_1, Y_1] + [X_2, Y_1] - Y_1(f)X_1$;

$$\begin{aligned} R(fX_1 + X_2, Y_1) &= f\nabla_{Y_1}\nabla_{X_1} + Y_1(f)\nabla_{X_1} + \nabla_{Y_1}\nabla_{X_2} - f\nabla_{X_1}\nabla_{Y_1} - \nabla_{X_2}\nabla_{Y_1} \\ &\quad + \nabla_{f[X_1, Y_1] + [X_2, Y_1] - Y_1(f)X_1} \\ &= f\nabla_{Y_1}\nabla_{X_1} + Y_1(f)\nabla_{X_1} + \nabla_{Y_1}\nabla_{X_2} - f\nabla_{X_1}\nabla_{Y_1} - \nabla_{X_2}\nabla_{Y_1} \\ &\quad + \nabla_{f[X_1, Y_1]} + \nabla_{[X_2, Y_1]} - \nabla_{Y_1(f)X_1} \\ &= f\nabla_{Y_1}\nabla_{X_1} + Y_1(f)\nabla_{X_1} - f\nabla_{X_1}\nabla_{Y_1} - \nabla_{X_2}\nabla_{Y_1} + \nabla_{Y_1}\nabla_{X_2} \\ &\quad + f\nabla_{[X_1, Y_1]} + \nabla_{[X_2, Y_1]} - Y_1(f)\nabla_{X_1} \\ &= f\nabla_{Y_1}\nabla_{X_1} - f\nabla_{X_1}\nabla_{Y_1} + f\nabla_{[X_1, Y_1]} + \nabla_{Y_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{Y_1} + \nabla_{[X_2, Y_1]} \\ R(fX_1 + X_2, Y_1) &= fR(X_1, Y_1) + R(X_2, Y_1) \end{aligned}$$

This shows relation (A.0.5). In this previous development, the sixth line comes by putting together the first, third, and fifth terms of the expression in line fifth, and cancel the second terms with the last one.
To show relation (A.0.6), we can see that $R(X_1, fY_1 + Y_2) = -R(fY_1 + Y_2, X_1)$, and using the previous result with $X_1 = X_2$, we get

$$R(X_1, fY_1 + Y_2) = -fR(Y_1, X_1) - R(Y_1, X_1)) = fR(X_1, Y_1) + R(X_1, Y_1).$$

So R is Bi-lineary in $\mathfrak{X}(M) \times \mathfrak{X}(M)$.

- Let $X, Y \in \mathfrak{X}(M)$ and $R(X, Y)$ a linear map from $\mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$.
  Let $Z_1, Z_2, Z \in \mathfrak{X}(M)$, and $f \in \mathcal{D}(M)$, we need to show that:

$$R(X, Y)(Z_1 + Z_2) = R(X, Y)(Z_1) + R(X, Y)(Z_2) \tag{A.0.7}$$
$$R(X, Y)(fZ) = fR(X, Y)(Z). \tag{A.0.8}$$

We have :

$$\begin{aligned} R(X, Y)(Z_1 + Z_2) &= \nabla_Y \nabla_X (Z_1 + Z_2) - \nabla_X \nabla_Y (Z_1 + Z_2) + \nabla_{[X, Y]} (Z_1 + Z_2) \\ &= \nabla_Y \nabla_X Z_1 - \nabla_X \nabla_Y Z_1 + \nabla_{[X, Y]} Z_1 \\ &\quad + \nabla_Y \nabla_X Z_2 - \nabla_X \nabla_Y Z_2 + \nabla_{[X, Y]} Z_2 \\ R(X, Y)(Z_1 + Z_2) &= R(X, Y)(Z_1) + R(X, Y)(Z_2). \end{aligned}$$

This shows (A.0.7);

$$R(X, Y)(fZ) = \nabla_Y \nabla_X (fZ) - \nabla_X \nabla_Y (fZ) + \nabla_{[X, Y]} (fZ).$$

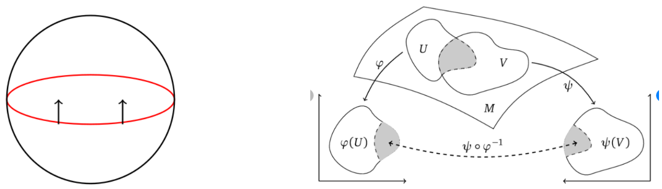
Since:

$$
\begin{aligned}
\nabla_Y \nabla_X (fZ) &= YX(f)Z + X(f)\nabla_Y Z + f\nabla_Y \nabla_X Z + Y(f)\nabla_X Z && \text{(A.0.9)}\\
\nabla_X \nabla_Y (fZ) &= XY(f)Z + Y(f)\nabla_X Z + X(f)\nabla_Y Z + f\nabla_X \nabla_Y Z && \text{(A.0.10)}\\
\nabla_{[X,Y]}(fZ) &= [X,Y](f)Z + f\nabla_{[X,Y]}(Z) && \text{(A.0.11)}\\
\text{(A.0.9)} - \text{(A.0.10)} &= [Y,X](f)Z + f(\nabla_Y \nabla_X - \nabla_X \nabla_Y)Z. && \text{(A.0.12)}
\end{aligned}
$$

Hence summing (A.0.12) with (A.0.11), we have :

$$
\begin{aligned}
R(X,Y)(fZ) &= [Y,X](f)Z + f(\nabla_Y \nabla_X - \nabla_X \nabla_Y)Z + [X,Y](f)Z + f\nabla_{[X,Y]}(Z)\\
&= -[X,Y](f)Z + f(\nabla_Y \nabla_X - \nabla_X \nabla_Y)Z + [X,Y](f)Z + f\nabla_{[X,Y]}(Z)\\
&\quad \text{Since the first and third terms will cancel in this previous expression;}\\
&= f(\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z)\\
R(X,Y)(fZ) &= fR(X,Y)(Z).
\end{aligned}
$$

This shows (A.0.8), and ends the proof of the proposition.



(a) Curve γ with the parallel field

(b) Transition map [10].

A.0.3 Corollary 3.2.16 .

Proof.

1. Let us prove that $\Gamma_{ij}^m = \Gamma_{ji}^m$.

   Since we are working with the Levi-Civita connection, using the first condition of Theorem 3.2.13, with $X = X_i, Y = X_j$ we have :

$$
\nabla_{X_i} X_j - \nabla_{X_j} X_i = [X_i, X_j]\,. \tag{A.0.13}
$$

   Let $f \in \mathcal{D}(M)$, and consider the restriction of f to U, evaluating the right hand side of (A.0.13), we have:

$$
\begin{aligned}
[X_i, X_j]\, f &= X_i X_j(f) - X_j X_i(f)\\
&= \frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_j}(p) - \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i}(p)\\
&= 0 \text{ ,since } f \in \mathcal{D}(M) \text{ we can swap the order of differentiation}\\
[X_i, X_j]\, f &= 0 \text{ , for all } f \implies [X_i, X_j] = 0.
\end{aligned}
$$

   (A.0.13), becomes:

$$
\nabla_{X_i} X_j - \nabla_{X_j} X_i = 0 \tag{A.0.14}
$$

$$(\text{A.0.14}) \implies \Gamma_{ij}^m X_m - \Gamma_{ji}^m X_m = 0$$
$$\implies \Gamma_{ij}^m = \Gamma_{ji}^m.$$

This ends the proof.

2. Using (3.2.9) of Lemma 3.2.14, with $X = X_i, Y = X_j, Z = X_l$, we have

$$2 < \nabla_{X_i} X_j, X_l >= X_i(< X_j, X_l >) + X_j(< X_i, X_l >) - X_l(< X_i, X_j >), \tag{A.0.15}$$

because $[X_i, X_l] = [X_j, X_l] = [X_l, X_i] = 0$.

$$\begin{aligned}
(\text{A.0.15}) \quad &\implies \quad 2 < \Gamma_{ij}^m X_m, X_l >= X_i(g_{jl}) + X_j(g_{il}) - X_l(g_{ij}),\\
&\implies \quad 2\Gamma_{ij}^m < X_m, X_l >= \frac{\partial}{\partial x_i}(g_{jl}) + \frac{\partial}{\partial x_j}(g_{il}) - \frac{\partial}{\partial x_l}(g_{ij}),\\
&\implies \quad 2\Gamma_{ij}^m g_{ml} = g_{jl,i} + g_{il,j} - g_{ij,l},\\
&\implies \quad \Gamma_{ij}^m = \frac{1}{2}(g_{ml})^{-1}\left(g_{jl,i} + g_{il,j} - g_{ij,l}\right),\\
&\implies \quad \Gamma_{ij}^m = \frac{1}{2}g^{ml}\left(g_{jl,i} + g_{il,j} - g_{ij,l}\right),
\end{aligned}$$

where g^{ml} are components of the inverse matrix associate to inner product. This ends the proof

□

A.0.4 Proof of the Lemma 3.2.4.

Proof. We are going to use Einstein summation convention. Let $p \in M$, consider the local coordinate around $p = (x_1, ..., x_n)$, and the local basis $(X_i)_{i=1,..,n}$ of T_pM, where $X_i = \frac{\partial}{\partial x_i}$. Taking $X, Y \in T_pM$, we can write $X = x^i X_i,\ Y = y^j X_j$, where x^i, y^j are differentiable functions, let us introduce the Christophel symbols by:

$$\nabla_{X_i} X_j = \Gamma_{ij}^k X_k. \tag{A.0.16}$$

where Γ_{ij}^k are differential function, which are components of the connection in the local basis, then we have

$$\begin{aligned}
\nabla_X Y &= \nabla_{x^i X_i} Y\\
&= x^i \nabla_{X_i} Y\\
&= x^i \nabla_{X_i}(y^j X_j)\\
&= x^i X_i(y^j) X_j + x^i y^j \nabla_{X_i} X_j\\
\nabla_X Y &= x^i \nabla_{X_i}(y^j) X_j + x^i y^j \Gamma_{ij}^k X_k \quad \text{using (A.0.16)}.
\end{aligned}$$

Since $x^i \nabla_{X_i}(y^j) = X(y^j)$

$$\begin{aligned}
\nabla_X Y &= X(y^j) X_j + x^i y^j \Gamma_{ij}^k X_k = X(y^k) X_k + \Gamma_{ij}^k X_k\\
\nabla_X Y &= \left(X(y^k) + x^i y^j \Gamma_{ij}^k\right) X_k.
\end{aligned} \tag{A.0.17}$$

Hence by (A.0.17), we see that $\nabla_X Y$ depend on the values of $X(y^k)(p)$ which is the derivative of y^k by X at the point p, and also $x^i(p),\ y^j(p)$. □

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