Optimal Control of a Pension Fund With Random Endowment

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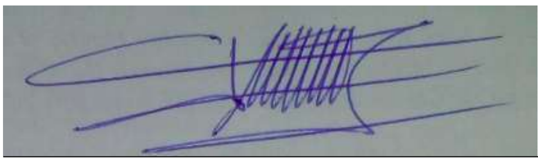
Abstract

This essay aims at studying a utility maximization or optimal investment problem in the presence of a stochastic endowment that cannot be traded in financial market. We use the dynamic programming approach to solve the optimization problem. We derive the Hamiltonian-Jacobi-Bellman equation for the value function for the control problem which is the non linear PDE. Furthermore, the techniques from the theory of viscosity solution and the homogeneity property of the value function are used to reduce the dimension of the HJB equation, which makes the problem accessible to numerical solutions. The main contribution of this project is to develop a finite difference method with implicit timestepping and non-equidistant grid to solve the resulting non-linear reduced HJB equation. We also derive the optimal investment strategy and discuss its asymptotic behaviour.

Keywords. Optimal control; HJB equation; stochastic endowment; viscosity solution; finite difference scheme.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



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1. Introduction

We study the problem of optimal control of a pension fund with random endowment. In [3], Chen et al. defined a pension fund as an investment product into which scheme members pay contributions in order to build up a lump sum to provide an income retirement. The government also pays back the income tax deducted from these contributions. In many cases, this is topped up with contributions from the scheme member's employer. There are two main types of pension schemes: defined contribution (DC) pension schemes and defined benefit pension schemes. Defined benefit pension schemes promise an income retirement based on your tenure at an employer and the wage you earn. DC pension plans has become very popular over the last year and are substituting defined benefit pension schemes. The later case is rigorously treated in [3].

Consider for example a person who receives random salaries during his working life and invests some initial capital and a fixed proportion of the salary into a portfolio such that the utility of the terminal wealth at the retirement age is maximized. In this example, the income is stochastic and will not in general be perfectly correlated with the traded assets in the market and hence making the market incomplete.

The objective of this work is to analyze the utility maximization or optimal investment problem of an economic agent under stochastic endowments for a finite time period.

The utility maximization problem of an economic agent by investment and/or consumption dates back to Merton in [13] and is further studied in [3, 5]. In the literature, there are many approaches to solve such problems. The first one is called the dynamic programming approach which requires the assumption of Markovianity on the state process and leads to the HJB equation developed in [2]. The second one is called dual approach where the assumption of Markovian asset prices can be relaxed is used in [4] to solve such problems under the assumption that the stochastic endowment is bounded. Here, the existence and uniqueness of an optimal control are proven, but an explicit representation of the optimal strategy is decomposed into a regular and a singular part, which is hard to characterize explicitly. In [12], the authors overcome the problem and relax the hypothesis of boundedness, by introducing the stochastic endowment as a new control variable. In [11], the approach developed in [12] is extended to the case of power utility functions, based on the martingale optimality principle and reduce the problem to the solution of a fully-coupled forward-backward stochastic differential equation, which is still not easy to solve.

In this work the problem is treated by proving, using analytic and, in particular, *viscosity solutions* techniques, that the value function of the stochastic control problem is a smooth solution of the associated HJB equation. Furthermore, the verification approach is used to make sure that given a smooth solution to the HJB equation, this candidate coincides with the value function. In addition, we prove analytically and confirm numerically that for large wealth, the optimal strategy approaches the original Merton ratio. The work is organized as follows. Chapter 2 describes the model formulation and how the HJB equation arises from the optimal control problem. In Chapter 3, we use the techniques from the theory of viscosity solution and the properties of the value function, in particular, the homogeneity property to reduce the dimension of the HJB equation associated to the optimal control problem. In Chapter 4 we study the asymptotic behaviour of the value function and the optimal strategy, as the initial investment approaches zero or infinity. Chapter 5 concludes the work and provides some perspectives for future research.

2. Model setup and optimal control problem

This chapter introduces the Hamiltonian-Jacobi-Bellman (HJB) equation and shows how it arises from optimal control problem. First of all, problem formulation is presented in section 2.1, afterwards optimal optimisation problem is presented in section 2.2, then the HJB equation is derived under strong assumption in section 2.3. The importance of this is that it will be used as part of the strategy for solving the optimization problem. Our primary sources for this chapter are [2, 5, 8, 14].

2.1 Problem formulation

To build the model, we follow the same idea as in Chen et al. [2].

We consider a financial market consisting of two assets: a riskless asset (a savings account S^0) and risky asset (a stock S^1).

Let T be a fixed finite period,

$r \in \mathbb{R}$ be a risk-free continuously compounded interest rate,

$\mu \in \mathbb{R}$ the average stock return (drift),

$\sigma > 0$ be the volatility,

S_t^0 be the bond price at time t and

S_t^1 be the stock price at time t. We assume that the two assets follow a Black-Scholes model:

$$\begin{aligned} dS_t^0 &= rS_t^0 dt \\ dS_t^1 &= \mu S_t^1 dt + \sigma S_t^1 dW_t^1, \end{aligned}$$

where W^1 is a Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$.

Assume that there is another process c with the stochastic dynamic driven by another Wiener process W^C on the same space correlated with W^1 with the correlation coefficient $\rho \in (-1, 1)$. Then there is a Wiener process W^2 independent of W^1 such that

$$W^C = \rho W^1 + bW^2,$$

where b is a constant. Since W^C is Wiener process and W^1 independent of W^2, we have

$$\begin{aligned} Var(W_t^C) = t &= Var(\rho W_t^1 + bW_t^2) \\ &= Var(\rho W_t^1) + Var(bW_t^2) \\ &= \rho^2 Var(W_t^1) + b^2 Var(W_t^2) \\ &= \rho^2 t + b^2 t. \end{aligned}$$

This implies that

$$1 = \rho^2 + b^2 \quad \Rightarrow \quad b = \pm\sqrt{1-\rho^2}.$$

Thus W^C can be written as

$$W^C = \rho W^1 + \sqrt{1-\rho^2}W^2.$$

The process c_t describes a random endowment or income, with the following dynamics:

$$\begin{aligned} dc_t &= \mu_C c_t dt + \sigma_C c_t dW_t^C \\ &= \mu_C c_t dt + \sigma_C c_t[\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2] \\ &= \mu_C c_t dt + \rho\sigma_C c_t dW_t^1 + \sigma_C c_t\sqrt{1-\rho^2}dW_t^2. \end{aligned}$$

Then given that at time t the initial endowment is $y \in \mathbb{R}_+$, we obtain

$$dc_s = \mu_C c_s ds + \rho\sigma_C c_s dW_s^1 + \sigma_C c_s\sqrt{1-\rho^2}dW_s^2, \quad c_t = y \tag{2.1.1}$$

where $\mu_C : [0,T] \longrightarrow \mathbb{R}$ and $\sigma_C : [0,T] \longrightarrow \mathbb{R}$ are deterministic right-continuous functions with left limit.

Portfolio: Let $X_0 > 0$ be the initial capital, θ_t^1 the number of stock at time $t \in [0,T]$, θ_t^0 the number of bonds at time $t \in [0,T]$ and X_t the Wealth of portfolio at time $t \in [0,T]$.

Assumption A_1: $X_t > 0$ almost surely, $t \in [0,T]$.

We assume that an agent invests at any time t a proportion $\pi_t = \frac{\theta_t^1 S_t^1}{X_t}$ of the wealth in the stock S^1 and the remaining $1-\pi_t = \frac{\theta_t^0 S_t^0}{X_t}$ in the bond S^0 with the interest rate r. Then

$$\theta_t^1 = \frac{X_t\pi_t}{S_t^1} \quad \text{and} \quad \theta_t^0 = \frac{X_t(1-\pi_t)}{S_t^0}.$$

In addition, we assume the random income is paid continuously at rate c_t. Here $\theta_t^i S_t^i$ is called the cash flow invested in the asset i at time t.

Definition 2.1.1. *A stochastic process $\pi = (\pi_t)_{t\in[0,T]}$ is called strategy.*

Special strategies:

- $\pi = 0$ pure bond strategy buys and holds bond.
- $\pi = 1$ pure stock strategy buys and holds stock.
- $\pi = \frac{1}{2}$ mixed strategy.

Assumption A_2. Self-financing condition.

The wealth process corresponding to the strategy π, $(X_t^\pi)_{t\in[0,T]}$ has the following dynamics:

$$\begin{aligned} dX_t^\pi &= \theta_t^0 dS_t^0 + \theta_t^1 dS_t^1 + c_t dt \\ &= \frac{X_t^\pi(1-\pi_t)}{S_t^0} rS_t^0 dt + \frac{X_t^\pi\pi_t}{S_t^1}[\mu S_t^1 dt + \sigma S_t^1 dW_t^1] + c_t dt \\ &= [X_t^\pi(r + \pi_t(\mu - r) + c_t]dt + X_t^\pi\pi_t\sigma dW_t^1. \end{aligned}$$

Hence, defining $\theta = \frac{\mu - r}{\sigma}$, for an initial wealth $x \in \mathbb{R}_+$ we obtain

$$dX_s^\pi = [X_s^\pi(r + \pi_s\sigma\theta) + c_s]ds + X_s^\pi\pi_s\sigma dW_s^1, \; X_t^\pi = x \tag{2.1.2}$$

Combining (2.1.1) and (2.1.2) we obtain the following model:

$$\begin{aligned} dX_s^\pi &= [X_s^\pi(r+\pi_s\sigma\theta)+c_s]ds + X_s^\pi\pi_s\sigma dW_s^1,\ X_t^\pi = x \\ dc_s &= \mu_C c_s ds + \rho\sigma_C c_s dW_s^1 + \sigma_C c_s\sqrt{1-\rho^2}dW_s^2,\ c_t = y \end{aligned} \tag{2.1.3}$$

Assumption A_3. For the investment strategy π and the endowment rate c, it holds:

- π and c are $\mathbb{F}$-progressively measurable processes with $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$.
- π admissible, that is, π takes values in a fixed closed convex subset K of $\mathbb{R}$,

$\pi_s \in K \Rightarrow \int_0^T |\pi_s|^2 ds < \infty$ and $X_t^\pi \geq 0$ for every $t \in [0,T]$.

We denote by $\mathcal{A}$ the set of all admissible strategies.

Remark 2.1.1. *The square-integrability condition ensures the existence and uniqueness of a solution for (2.1.3)*

2.2 Optimization problem: Optimal terminal wealth

Find an optimal investment strategy π^* such that

$$v(t,x,y) = J^{\pi^*}(t,x,y)$$

where

$$v(t,x,y) = \sup_{\pi\in\mathcal{A}_1} J^\pi(t,x,y) \tag{2.2.1}$$

with

$$J^\pi(t,x,y) = \mathbb{E}[U(X_T^{\pi,t,x,y})]$$

and

$$\mathcal{A}_1 = \{\pi \in \mathcal{A}(t,x,y) : \mathbb{E}[U(X_T^{\pi,t,x,y})] < \infty\}.$$

We call

J^π the performance criterion or reward function,

v the value function of the utility maximization problem,

$\mathcal{A}_1$ the set of admissible strategies or admissibility set and

$U : \mathbb{R} \Rightarrow \mathbb{R}_+$ is a constant relative risk aversion power utility function given by

$$U(x) = \frac{x^\gamma}{\gamma},$$

for a risk reversion parameter $\gamma < 1,\ \gamma \neq 0$. The properties of the utility are well-studied in [10, Chapter 7]. Here we recall some of them.

Properties 2.2.1. • *U is strictly increasing, strictly convave on* $(0,+\infty)$ *and twice continuously differentiable*

- $U'(0) := \lim_{x\to 0} U'(x) = \infty,\quad U'(\infty) := \lim_{x\to\infty} U'(x) = 0$ *(Inada condition)*

Interpretation

Strictly increasing means that the investor prefers more to less wealth. The strictly concave decreases the slope of $U(x)$ $(U''(x) < 0)$.

$\lim_{x\to\infty} U'(x) = 0$ can be seen as the saturation effect and $\lim_{x\to 0} U'(x) = \infty$ (infinite slope at $x = 0$) means that small money is very much better than nothing at all .

Remark 2.2.1. • *The power utility belongs to the class of* constant relative risk aversion *utility function because the relative risk aversion*

$$RRA(x) = -\frac{xU''(x)}{U'(x)} = 1 - \gamma = const.$$

• *The use of the power utility is well-motivated economically, since the long-run behavior of the economy suggests that the long run risk aversion cannot strongly depend on wealth.*

We are taking as controlled process $Y^\pi = (X^\pi, c)$, and the notation $X^{\pi,t,x,y}$ stands for the first coordinate of the process Y^π starting from the point (x, y), respectively the initial wealth and the initial endowment, at time t.

Note that the process c does not depend on the control π, nor on the initial wealth. Therefore, we can write $c^{t,y}$ for $c^{\pi,t,x,y}$.

2.3 Derivation of Hamilton-Jacobi-Bellman (HJB) equation

To derive the HJB equation for optimization problem in section 2.2, we follow the same idea as in [8] in the case of optimal terminal wealth and we apply the well-known result in stochastic control, see [14, Chapter 3].

Given an optimal control problem in section 2.2, we have two natural questions to answer:

(a) Does the optimal strategy exist?

(b) Given that an optimal control exists, how do we find it?

The main idea is to transform our original problem into a partial differential equation (PDE) known as the Hamilton-Jacobi-Bellman equation. The control problem is then shown to be equivalent to the problem of finding a solution to the HJB equation. We are now going to describe the transformation procedure, and for that purpose we assume the following:

1) there exists an optimal control π^*,

2) the optimal value function is regular ($v \in C^{1,2}$) and the dynamics of the controlled process is given by

$$dY_s^\pi = \mu(X_s^\pi, c_s, \pi_s)dt + \sigma(X_s^\pi, c_s, \pi_s)dW_s. \tag{2.3.1}$$

Let $G(x, y, \pi_t) = \sigma(x, y, \pi_t) \cdot \sigma(x, y, \pi_t)^T$ and let K be a closed convex subset of $\mathbb{R}$, we have the following:

Theorem 2.3.1. *Under the above assumptions, the value function (2.2.1) satisfies the HJB equation*

$$-\frac{\partial v(t,x,y)}{\partial t} = \sup_{p\in K}\{\mathcal{L}^p v(t,x,y)\}, \tag{2.3.2}$$

where $\mathcal{L}^p$ is the generator of the controlled process (2.3.1) given by

$$\mathcal{L}^p = \sum_{i=1}^{2}\mu^i(x,y,p)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{2}G^{ij}(x,y,p)\frac{\partial^2}{\partial x_i\partial x_j}, \tag{2.3.3}$$

with terminal condition

$$v(T,x,y) = U(x), \quad \forall\ (x,y)\in(0,\infty)\times(0,\infty).$$

Proof. To prove Theorem 2.3.1, we

- fix $(t,x,y)\in[0,T]\times\mathbb{R}\times\mathbb{R}$,
- choose $h>0$ such that $t+h<T$,
- choose an arbitrary control $\pi\in\mathcal{A}_1$.

Define a control $\widehat{\pi}$ by

$$\widehat{\pi}(s,x,y) = \begin{cases}\pi(s,x,y), & (s,x,y)\in[t,t+h]\times\mathbb{R}\times\mathbb{R}\\ \pi^*(s,x,y), & (s,x,y)\in(t+h,T]\times\mathbb{R}\times\mathbb{R}\end{cases}$$

Expected utility for strategy π^*: This is trivial, since by definition the utility is the optimal one given by

$$J^{\pi^*}(t,x,y) = v(t,x,y).$$

Expected utility for strategy $\widehat{\pi}$: We divide the time interval $[t,T]$ into two parts, the intervals $[t,t+h]$ and $(t+h,T]$ respectively.

(i) The expected utility, using $\widehat{\pi}$ for the interval $[t,t+h)$, is zero since we are dealing with an optimal terminal wealth.

(ii) In the interval $[t+h,T]$ we observe that at time $t+h$ we will be in the state (X^{π}_{t+h},c_{t+h}). Since, by definition, we will use the optimal strategy π^* during the entire interval $[t+h,T]$ we see that the remaining expected utility at time $t+h$ is given by $v(t+h,X^{\pi}_{t+h},c_{t+h})$. Thus the expected utility over the interval $[t+h,T]$, conditional on the fact that at time t we are in state (x,y), is given by

$$\mathbb{E}_{t,x,y}[v(t+h,X^{\pi}_{t+h},c_{t+h})].$$

Then the total expected utility for strategy $\widehat{\pi}$ is

$$J^{\widehat{\pi}}(t,x,y) = \mathbb{E}_{t,x,y}[v(t+h,X^{\pi}_{t+h},c_{t+h})].$$

Comparing the strategies: Since π^* is optimal we have

$$J^{\pi^*}(t,x,y) \geq J^{\widehat{\pi}}(t,x,y) \ \ (with\ equality\ if\ \widehat{\pi}=\pi^*).$$

This implies

$$v(t,x,y) \geq \mathbb{E}_{t,x,y}[v(t+h, X^{\pi}_{t+h}, c_{t+h})]. \tag{2.3.4}$$

We apply the Dynkin's formula to obtain

$$\mathbb{E}_{t,x,y}[v(t+h, X^{\pi}_{t+h}, c_{t+h})] = v(t,x,y) + \mathbb{E}_{t,x,y}\left[\int_t^{t+h}\left(\frac{\partial v}{\partial t} + \mathcal{L}^{\pi_s} v\right)(s, X^{\pi}_s, c_s)ds\right],$$

where $\mathcal{L}^{\pi}$ is the generator of the state controlled process given by (2.3.3).

Substituting back in (2.3.4) yields

$$\mathbb{E}_{t,x,y}\left[\int_t^{t+h}\left(\frac{\partial v}{\partial t} + \mathcal{L}^{\pi_s} v\right)(s, X^{\pi}_s, c_s)ds\right] \leq 0.$$

Going to the limit: Now we divide by h, move h within the expectation and let h tend to zero. Assuming enough regularity to allow us to take the limit within the expectation, using the fundamental theorem of integral calculus $\left(\frac{1}{h}\int_t^{t+h} f(s)ds \longrightarrow f(t) \text{ as } h \longrightarrow 0\right)$ and recalling that $X^{\pi}_t = x$ and $c_t = y$ we get

$$\frac{\partial v(t,x,y)}{\partial t} + \mathcal{L}^{p} v(t,x,y) \leq 0,$$

which is equivalent to

$$-\frac{\partial v(t,x,y)}{\partial t} - \mathcal{L}^{p} v(t,x,y) \geq 0. \tag{2.3.5}$$

On the other hand, suppose that π^* is an optimal control. Then (2.3.4) becomes

$$v(t,x,y) = \mathbb{E}_{t,x,y}[v(t+h, X^{\pi^*}_{t+h}, c_{t+h})].$$

Using the same argument as above, we obtain

$$-\frac{\partial v(t,x,y)}{\partial t} - \mathcal{L}^{p^*} v(t,x,y) = 0. \tag{2.3.6}$$

Combining (2.3.5) and (2.3.6), we suggest that v should satisfy the PDE called Hamilton-Jacobi-Bellman equation

$$-\frac{\partial v(t,x,y)}{\partial t} = \sup_{p\in K}\{\mathcal{L}^{p} v(t,x,y)\}. \tag{2.3.7}$$

□

Lemma 2.3.1. *Given the initial wealth x and the initial endowment y at time t, the HJB equation (2.3.2) becomes*

$$-v_t = \sup_{p\in K}\{[x(r+p\sigma\theta)+y]v_x + \mu_C(t)yv_y + \frac{1}{2}(xp\sigma)^2 v_{xx} + \frac{1}{2}(y\sigma_C(t))^2 v_{yy} + \rho\sigma\sigma_C(t)yxpv_{xy}\}, \tag{2.3.8}$$

with the terminal condition

$$v(T,x,y) = U(x), \quad \forall\ (x,y) \in (0,\infty)\times(0,\infty).$$

Proof. We recall that given the initial wealth x and the initial endowment y at time t, the dynamics of the controlled process is given by

$$dc_s = \mu_C c_s ds + \rho\sigma_C c_s dW_s^1 + \sigma_C c_s\sqrt{1-\rho^2}dW_s^2, c_t = y$$
$$dX_s^\pi = [X_s^\pi(r + \pi_s\sigma\theta) + c_s]ds + X_s^\pi\pi_s\sigma dW_s^1, X_t^\pi = x$$

We recall also that the dynamics of the controlled process $Y^\pi = (X^\pi, c)$ is given by

$$dY_s^\pi = \mu(X_s^\pi, c_s, \pi_s)dt + \sigma(X_s^\pi, c_s, \pi_s)dW_s,$$

where

$$\mu(x,y,p) = \begin{pmatrix} x(p\sigma\theta + r) + y \\ \mu_C(t)y \end{pmatrix}, \quad \sigma(x,y,p) = \begin{pmatrix} xp\sigma & 0 \\ \rho\sigma_C(t)y & y\sigma_C(t)\sqrt{1-\rho^2} \end{pmatrix} \text{ and } W = \begin{pmatrix} W^1 \\ W^2 \end{pmatrix}.$$

We hence have

$$G = G(x,y,p) = \sigma(x,y,p).\sigma(x,y,p)^T = \begin{pmatrix} xp\sigma & 0 \\ \rho\sigma_C(t)y & y\sigma_C(t)\sqrt{1-\rho^2} \end{pmatrix}.\begin{pmatrix} xp\sigma & \rho\sigma_C(t)y \\ 0 & y\sigma_C(t)\sqrt{1-\rho^2} \end{pmatrix}.$$

Thus

$$G(x,y,p) = \begin{pmatrix} (xp\sigma)^2 & \rho\sigma\sigma_C(t)xyp \\ \rho\sigma\sigma_C(t)yxp & (y\sigma_C(t))^2 \end{pmatrix}.$$

The generator of Y^π is given by (2.3.3) becomes

$$\mathcal{L}^p v(t,x,y) = \sum_{i=1}^{2}\mu^i(x,y,p)\frac{\partial v}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{2}G^{ij}\frac{\partial^2 v}{\partial x_i\partial x_j}$$

$$= \mu^1(x,y,p)\frac{\partial v}{\partial x} + \mu^2(x,y,p)\frac{\partial v}{\partial y} + \frac{1}{2}G^{11}\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}G^{22}\frac{\partial^2 v}{\partial y^2} + \frac{1}{2}G^{12}\frac{\partial^2 v}{\partial x\partial y} + \frac{1}{2}G^{21}\frac{\partial^2 v}{\partial y\partial x}$$

$$= [x(r + p\sigma\theta) + y]\frac{\partial v}{\partial x} + \mu_C(t)y\frac{\partial v}{\partial y} + \frac{1}{2}(xp\sigma)^2\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}(y\sigma_C(t))^2\frac{\partial^2 v}{\partial y^2} + \rho\sigma\sigma_C(t)yxp\frac{\partial^2 v}{\partial x\partial y},$$

which can be written as

$$\mathcal{L}^p v(t,x,y) = [x(r + p\sigma\theta) + y]v_x + \mu_C(t)yv_y + \frac{1}{2}(xp\sigma)^2 v_{xx} + \frac{1}{2}(y\sigma_C(t))^2 v_{yy} + \rho\sigma\sigma_C(t)yxpv_{xy}.$$

Hence the Hamilton-Jacobi-Bellman equation (2.3.2) becomes

$$-v_t = \sup_{p\in K}\{[x(r + p\sigma\theta) + y]v_x + \mu_C(t)yv_y + \frac{1}{2}(xp\sigma)^2 v_{xx} + \frac{1}{2}(y\sigma_C(t))^2 v_{yy} + \rho\sigma\sigma_C(t)yxpv_{xy}\}$$

with terminal condition (2.3.9)

$v(T,x,y) = U(x), \quad \forall\ (x,y).$

□

3. Hamilton-Jacobi-Bellman equation

The HJB approach for solving the optimal control problem consists of solving the HJB equation in order to obtain the value function. But there can be many solutions or no solution for equation (2.3.9). That is why the notion of viscosity solution of the HJB equation is needed. The purpose of this chapter is to use techniques from the theory of viscosity solutions to prove that the value function is the unique viscosity solution to the HJB equation (2.3.9) in Section 3.2 and then using this to reduce the dimension of our HJB equation in Section 3.3. In Section 3.4, the verification approach is used to confirm that given a smooth solution to the HJB equation, this candidate coincides with the value function. We begin with some useful properties of the value function.

3.1 Some properties of the value function

All results in this section are obtained by extending the methods in [14, Chapter 3] in dimension 1 to the dimension two case since we are dealing with the problem of optimization of terminal wealth in the presence of a random endowment.

Proposition 3.1.1. *The value function $v(t,x,y)$ is increasing, concave, and hence continuous in the second variable in the interior of the domain.*

Proof. • We want to show that the value function is increasing.

Fix $0 < x_1 < x_2,\ 0 < t < T$ and $y > 0$.

To ease the notation we set $X^1 = X^{\pi,t,x_1,y}$ and $X^2 = X^{\pi,t,x_2,y}$. Let $Z_s = X_s^2 - X_s^1$ for $s > t$ Then $(Z_s)_{s>0}$ satisfies the following:

$$dZ_s = Z_s[(\pi_s\sigma\theta + r)dt + \pi_s\sigma dW_s^1],\ Z_t = x_2 - x_1 > 0.$$

Indeed,

$$\begin{aligned} dZ_s &= [X_s^2(\pi_s\sigma\theta + r) + c_s]dt + X_s^2\pi_s\sigma dW_s^1 - [X_s^1(\pi_s\sigma\theta + r) + c_s]dt + X_s^1\pi_s\sigma dW_s^1 \\ &= (X_s^2 - X_s^2)[(\pi_s\sigma\theta + r)dt + \pi_s\sigma dW_s^1] \\ &= Z_s[(\pi_s\sigma\theta + r)dt + \pi_s\sigma dW_s^1] \end{aligned}$$

and at time t , $Z_t = X_t^2 - X_t^1 = x_2 - x_1 > 0$.

Then $Z_s \geq 0$ for all $s > t$, this implies $X_s^2 \geq X_s^1$. Using the increasing property of the utility function we have that for all $\pi \in \mathcal{A}_1$

$$U(X_T^1) \leq U(X_T^2).$$

Then for all $\pi \in \mathcal{A}_1$

$$\mathbb{E}[U(X_T^1)] \leq \mathbb{E}[U(X_T^2)] \leq v(t,x_2,y).$$

Thus

$$\sup_{\pi\in\mathcal{A}_1} \mathbb{E}[U(X_T^1)] \leq v(t,x_2,y).$$

Hence

$$v(t, x_1, y) \le v(t, x_2, y).$$

This shows that the value function is increasing in the second variable.

• We want to show that v is concave.

Fix again $0 < t < T,\ x_2,\ x_1 > 0$ and $\pi^1,\ \pi^2 \in \mathcal{A}_1$ two controls and $\lambda \in [0,1]$. To ease the notation we set $X^1 := X^{\pi^1,t,x_1,y}$ and $X^2 := X^{\pi^2,t,x_1,y}$. $X^i, i = 1,2$ is the process starting at time t from $x_i,\ i = 1,2$ and controlled by $\pi^i,\ i = 1,2$.

Define

$$x^\lambda := \lambda x^1 + (1-\lambda)x^2, \quad X^\lambda := \lambda X^1 + (1-\lambda)X^2 \text{ and } \pi_s^\lambda := \frac{\lambda X_s^1 \pi_s^1 + (1-\lambda)X_s^2\pi_s^2}{\lambda X_s^1 + (1-\lambda)X_s^2}.$$

Then $\pi^\lambda \in \mathcal{A}_1$ since $\mathcal{A}_1$ is a closed convex set and $\pi^1,\ \pi^2 \in \mathcal{A}_1$.

Moreover, from the linear dynamic of the wealth process, we see that $(X_s^\lambda)_{s>t}$ is governed by:

$$dX_s^\lambda = [X_s^\lambda(\pi_s^\lambda \sigma\theta + r) + c_s^{t,y}]ds + X_s^\lambda \pi_s^\lambda \sigma dW_s^1,\ X_t^\lambda = x^\lambda.$$

Indeed,

$$\begin{aligned}
dX_s^\lambda &= \lambda dX_s^1 + (1-\lambda)dX_s^2 \\
&= \lambda\left\{[X_s^1(\pi_s^1\sigma\theta + r) + c_s^{t,y}]dt + X_s^2\pi_s^1\sigma dW_s^1\right\} + (1-\lambda)\left\{[X_s^2(\pi_s^2\sigma\theta + r) + c_s^{t,y}]dt + X_s^1\pi_s^2\sigma dW_s^1\right\} \\
&= \left\{[\lambda X_s^1\pi_s^1 + (1-\lambda)X_s^2\pi_s^2]\sigma\theta + r[\lambda X_s^1 + (1-\lambda)X_s^2]\right\}dt + c_s^{t,y}dt + [\lambda X_s^1\pi_s^1 + (1-\lambda)X_s^2\pi_s^2]\sigma dW_s^1 \\
&= (X_s^\lambda\pi_s^\lambda\sigma\theta + X_s^\lambda r)ds + c_s^{t,y}ds + X_s^\lambda\pi_s^\lambda\sigma dW_s^1 \\
&= [X_s^\lambda(\pi_s^\lambda\sigma\theta + r) + c_s^{t,y}]ds + X_s^\lambda\pi_s^\lambda\sigma dW_s^1
\end{aligned}$$

and $X_t^\lambda := \lambda X_t^1 + (1-\lambda)X_t^2 = \lambda x^1 + (1-\lambda)x^2$.

This shows that X^λ is the wealth process starting at time t form $x^\lambda := \lambda x^1 + (1-\lambda)x^2$ and controlled by π^λ.

By the concavity of the utility function U, we have that

$$U(\lambda X_T^1 + (1-\lambda)X_T^2) \ge \lambda U(X_T^1) + (1-\lambda)U(X_T^2).$$

Taking the expectation in both side of the inequality we have for all $\pi^1,\ \pi^2 \in \mathcal{A}_1$

$$v(t, \lambda x^1 + (1-\lambda)x^2, y) \ge \mathbb{E}[U(\lambda X_T^1 + (1-\lambda)X_T^2)] \ge \lambda\mathbb{E}[U(X_T^1)] + (1-\lambda)\mathbb{E}[U(X_T^2)].$$

Thus

$$v(t, \lambda x^1 + (1-\lambda)x^2, y) \ge \lambda \sup_{\pi\in\mathcal{A}_1} \mathbb{E}[U(X_T^1)] + (1-\lambda) \sup_{\pi\in\mathcal{A}_1} \mathbb{E}[U(X_T^2)].$$

We deduce that

$$v(t, \lambda x^1 + (1-\lambda)x^2, y) \ge \lambda v(t, x^1, y) + (1-\lambda)v(t, x^2, y).$$

This shows that the value function is concave. □

Definition 3.1.1. *A function f is homogeneous in (x, y) with degree k if and only if*

$$f(tx, ty) = t^k f(x, y) \quad \forall t > 0.$$

Remark 3.1.1. *If f is homogeneous with degree zero the can write*

$$f(x, y) = f\left(y \cdot \frac{x}{y}, y \cdot 1\right) = f\left(\frac{x}{y}, 1\right) = g\left(\frac{x}{y}\right),$$

where g is a function of one variable given by $g(z) = f(z, 1)$.

This motivates the following:

Lemma 3.1.1. *The value function (2.2.1) is homogeneous in (x, y) with degree γ, and therefore there exists a function $u : [0, T] \times (0, +\infty) \longrightarrow \mathbb{R}$ such that v can be represented in a separable form as*

$$v(t, x, y) = y^\gamma u\left(t, \frac{x}{y}\right).$$

To prove this lemma we use the following.

Proposition 3.1.2. *Under the assumption of the model (2.1.3), for every fixed strategy π, the explicit solution of equation (2.1.2) is given as:*

$$X_s^{t,x,y} = \left(x + \int_s^t \frac{c_u^{t,y}}{Z_u} du\right) Z_s, \tag{3.1.1}$$

where Z is a stochastic exponential factor given as:

$$Z_s = \exp\left\{\int_t^s \left[(\pi_u \sigma\theta + r) - \frac{1}{2}(\pi_u\sigma)^2\right] du + \int_t^s \sigma\pi_u dW_u^1\right\}.$$

Proof of proposition 3.1.2. Let P be a continuous semi-martingale with $P_0 = 0$, we define the stochastic exponential of P_s, written $\mathcal{E}(P)s$ as (see [15, Chapter 1]), which is the (unique) semi-martingale Z which is a solution of

$$Z_s = 1 + \int_0^s Z_t dP_t,$$

given as

$$\mathcal{E}(P)_s := \exp\left\{P_s - \frac{1}{2}\langle P\rangle_s\right\}.$$

We have now

$$\begin{aligned}
Z_s &= \exp\left\{\int_t^s \left[(\pi_u\sigma\theta + r) - \frac{1}{2}(\pi_u\sigma)^2\right] du + \int_t^s \sigma\pi_u dW_u^1\right\} \\
&= \exp\left\{\int_t^s (\pi_u\sigma\theta + r) du + \int_t^s \sigma\pi_u dW_u^1 - \frac{1}{2}\int_t^s (\pi_u\sigma)^2 du\right\} \\
&= \exp\left\{P_s - \frac{1}{2}\langle P\rangle_s\right\} \\
&= \mathcal{E}(P)_s,
\end{aligned}$$

where $P_s = \int_t^s (\pi_u \sigma\theta + r)du + \int_t^s \sigma\pi_u dW_u^1$. P is clearly a continuous semi-martingale starting at time t from $P_t = 0$.

Recall that, for a fixed strategy π, the wealth process $X = X^\pi$ has the following dynamics:

$$dX_s = [X_s(r + \pi_s\sigma\theta) + c_s]ds + X_s\pi_s\sigma dW_s^1, \ X_t^s = x$$

which can be written in terms of P_s as follows:

$$\begin{aligned} dX_s &= [X_s(r + \pi_s\sigma\theta) + c_s^{t,y}]ds + X_s\pi_s\sigma dW_s^1 \\ &= c_s^{t,y}ds + X_s dP_s. \end{aligned}$$

This implies that

$$X_s = X_t + \int_t^s c_u^{t,y}du + \int_t^s X_u dP_u,$$

so that

$$X_s = H_s + \int_t^s X_u dP_u, \tag{3.1.2}$$

where $H_s = x + \int_t^s c_u^{t,y}du$ is an adapted process with continuous paths of finite variation started at time t from $H_t = x$.

We apply the result in [15, Theorem 52], to obtain the explicit solution of (3.1.2) given as:

$$X_s^{t,x,y} = \mathcal{E}(P)_s \left\{ H_t + \int_t^s \frac{1}{\mathcal{E}(P)_s} d(H_u - \langle H, P\rangle_u) \right\}.$$

Since a càdlàg process H is of finite variation, we have $\langle H, P\rangle_u = 0$ and using the fact that $H_t = x, \ \mathcal{E}(P)_s = Z_s$ and $dH_u = c_u^{t,y}du$ we hence have

$$X_s^{t,x,y} = \left(x + \int_s^t \frac{c_u^{t,y}}{Z_u}du \right) Z_s.$$

□

Proof of Lemma 3.1.1. Recall that the random endowment has the following dynamics:

$$dc_t = \mu_C c_t dt + \sigma_C c_t dW_t^C$$

We apply the Itô formula to the function $\log x$ to obtain

$$c_s^{t,y} = c_t \exp\left\{ \int_t^s \left[\mu_C(u) - \frac{1}{2}\sigma_C^2(u) \right] du + \int_t^s \sigma_C(u) dW_u^C \right\} = y\mathcal{E}(Q)_s,$$

where $Q_s = \int_t^s \mu_C(u)du + \int_t^s \sigma_C(u)dW_u^C$.

It clear that the random endowment c is linear with respect to the initial value y. This means that for all $k > 0, \quad c_s^{t,ky} = kc_s^{t,y}$.

From section (2.2) we have that

$$v(t,kx,xy) = \sup_{\pi\in\mathcal{A}_1} \mathbb{E}\left[\frac{(X^{\pi,t,kx,ky})^\gamma}{\gamma}\right].$$

But from equation (3.1.1) we have that

$$X^{\pi,t,kx,ky} = \left(kx + \int_s^t \frac{c_u^{t,ky}}{Z_u}du\right) Z_s = \left(kx + \int_s^t \frac{kc_u^{t,y}}{Z_u}du\right) Z_s = k\left(x + \int_s^t \frac{c_u^{t,y}}{Z_u}du\right) Z_s = kX^{\pi,t,x,y}.$$

Hence

$$\begin{aligned} v(t,kx,xy) &= \sup_{\pi\in\mathcal{A}_1} \mathbb{E}\left[\frac{(kX^{\pi,t,x,y})^\gamma}{\gamma}\right] \\ &= k^\gamma \sup_{\pi\in\mathcal{A}_1} \mathbb{E}\left[\frac{(X^{\pi,t,x,y})^\gamma}{\gamma}\right] \\ &= k^\gamma v(t,x,y). \end{aligned}$$

This shows that the value function is homogeneous in (x,y) with degree γ.

This means that we can define a function $u : [0,T]\times(0,+\infty) \longrightarrow \mathbb{R}$ by $u(t,z) = v(t,z,1)$ and then, for every $y>0$, we will have that

$$v(t,x,y) = v\left(t, y\cdot\frac{x}{y}, y\cdot 1\right) = y^\gamma v\left(t,\frac{x}{y},1\right) = y^\gamma u\left(t,\frac{x}{y}\right).$$

□

3.2 Viscosity solution of the Hamiltonian-Jacobi-Bellman equation

In this section we analyze the HJB equation (2.3.9), using results from the theory of viscosity solutions. In particular, we show that the value function v is the unique viscosity solution of (2.3.9). We begin by defining the notion of *viscosity solution* as given in [14, Chapter 4]. There are a number of equivalent ways of defining viscosity solutions for parabolic PDE. In [14, Chapter 4], the notion of viscosity solutions is treated in detail, however for this work, it will be more helpful to use the definition from [5], where a less restrictive condition is placed on the auxiliary notions of a viscosity subsolution and supersolution. To be specific, subsolutions and supersolutions will be allowed to be semi-continuous, whereas in [5], these are taken to be continuous.

In the sequel we will use the following:

Notation 3.2.1. *Let $\mathcal{O} = [0,T]\times(0,+\infty)\times(0,+\infty)$ be a domain. Let $\varphi \in C^{1,2}(\mathcal{O})$, then we denote by φ_t the partial derivative with respect to t, $D\varphi = (\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y})^T \in \mathbb{R}^2$ the gradient of φ and $D^2\varphi \in S_2$ the Hessian matrix of φ, where S_2 is the space of 2-dimensional symmetric matrices.*

Definition 3.2.1. *Let $F : \mathcal{O}\times\mathbb{R}\times\mathbb{R}^2\times S_2 \longrightarrow \mathbb{R}$ be a continuous function. According to [14, Chapter 4] the function is called parabolic if for all $(t,x,y,q,p,M) \in \mathcal{O}\times\mathbb{R}\times\mathbb{R}^2\times S_2$ and $\widehat{q}\in\mathbb{R}$*

$$q \le \widehat{q} \Longrightarrow F(t,x,y,q,p,M) \ge F(t,x,y,\widehat{q},p,M).$$

Using the above notation we can write the HJB equation (2.3.9) as

$$F(t,x,y,s,q,p,M) = -q + \sup_{\pi_t \in K} \{[-yp_1 - \mu_C(t)yp_2 - (r + \pi_t\sigma\theta)p_1 - \frac{1}{2}(x\pi_t\sigma)^2 M_{11} - \frac{1}{2}(y\sigma_C(t))^2 M_{22} + \rho\sigma\sigma_C(t)xy\pi_t M_{12}\}.$$

So, the HJB equation (2.3.9) can be written in the compacted form

$$F(t,x,y,v(t,x,y),v_t(t,x,y),Dv(t,x,y),D^2v(t,x,y)) = F(t,x,y,s,q,p,M) = 0. \tag{3.2.1}$$

Definition 3.2.2. *Given a locally bounded function $w : \mathcal{O} \longrightarrow \mathbb{R}$, we define*

1) its upper-semicontinuous envelope

$$w^*(\overline{x}) = \limsup_{x \longrightarrow \overline{x}} w(x),$$

2) its lower-semicontinuous envelope

$$w_*(\overline{x}) = \liminf_{x \longrightarrow \overline{x}} w(x).$$

We recall that w is continuous if and only if $w = w_* = w^*$ on $\mathcal{O}$.

Definition 3.2.3 (viscosity solution)**.** *Let $w : \mathcal{O} \longrightarrow \mathbb{R}$ be locally bounded.*

i) w is a (discontinuous) viscosity subsolution of (3.2.1) on $\mathcal{O}$ if

$$F(\bar{t},\overline{x},\overline{y},w^*(\bar{t},\overline{x},\overline{y}),\varphi_t(\bar{t},\overline{x},\overline{y}),D\varphi(\bar{t},\overline{x},\overline{y}),D^2\varphi(\bar{t},\overline{x},\overline{y})) \leq 0,$$

for all $(\bar{t},\overline{x},\overline{y}) \in \mathcal{O}$ and for all $\varphi \in C^{1,2}(\mathcal{O})$ such that $(w^* - \varphi)(\bar{t},\overline{x},\overline{y}) = \max_{(t,x,y)\in\mathcal{O}}(w^* - \varphi)(t,x,y)$.

ii) w is a (discontinuous) viscosity supersolution of (3.2.1) on $\mathcal{O}$ if

$$F(\bar{t},\overline{x},\overline{y},w_*(\bar{t},\overline{x},\overline{y}),\varphi_t(\bar{t},\overline{x},\overline{y}),D\varphi(\bar{t},\overline{x},\overline{y}),D^2\varphi(\bar{t},\overline{x},\overline{y})) \geq 0,$$

for all $(\bar{t},\overline{x},\overline{y}) \in \mathcal{O}$ and for all $\varphi \in C^{1,2}(\mathcal{O})$ such that $(w_* - \varphi)(\bar{t},\overline{x},\overline{y}) = \min_{(t,x,y)\in\mathcal{O}}(w_* - \varphi)(t,x,y)$.

iii) We say that w is a (discontinuous) viscosity solution of (3.2.1) on $\mathcal{O}$ if it is both a subsolution and supersolution of (3.2.1) on $\mathcal{O}$.

Theorem 3.2.1. *The value function (2.2.1) is a viscosity solution of the HJB equation (3.2.1) associated to the optimization problem.*

Proof. See appendix A □

3.3 Reduced form of Hamilton-Jacobi-Bellman equation

To reduce the dimension of HJB equation we use the homogeneity property of the value function and technique from the theory of viscosity solutions. This technique is used already in [5] in the case of optimal consumption problem and in [2] in the case of optimal investment with time-varying stochastic endowments.

Theorem 3.3.1 (Viscosity solution for reduced equation). *1) For a fixed initial random endowment $\overline{y} > 0$, define $\overline{u}(t,z) = v(t,z,\overline{y})$. Then $\overline{u} : (0,T] \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a viscosity solution of the reduced PDE with parameter $\overline{y}$:*

$$\overline{u}_t + \overline{y}\overline{u}_z + \mu_C(t)[\gamma\overline{u} - z\overline{u}_z] + \frac{1}{2}\sigma_C^2(t)[\gamma(\gamma-1)\overline{u} - 2(\gamma-1)z\overline{u}_z + z^2\overline{u}_{zz}]+$$
$$\sup_{\pi\in\mathcal{A}_1}\left\{(\pi\sigma\theta + r)z\overline{u}_z + \frac{1}{2}(\pi\sigma)^2 z^2\overline{u}_{zz} + \rho\sigma_C(t)\sigma\pi(\gamma-1)z\overline{u}_z - \rho\sigma_C(t)\sigma\pi z^2\overline{u}_{zz}\right\} = 0, \quad (3.3.1)$$
$$\overline{u}(T,z) = \frac{z^\gamma}{\gamma} \quad \forall z \geq 0.$$

In particular, for $\overline{y} = 1$

$$u_t + u_z + \mu_C(t)[\gamma u - zu_z] + \frac{1}{2}\sigma_C^2(t)[\gamma(\gamma-1)u - 2(\gamma-1)zu_z + z^2u_{zz}]+$$
$$\sup_{\pi\in\mathcal{A}_1}\left\{(\pi\sigma\theta + r)zu_z + \frac{1}{2}(\pi\sigma)^2 z^2u_{zz} + \rho\sigma_C(t)\sigma\pi(\gamma-1)zu_z - \rho\sigma_C(t)\sigma\pi z^2u_{zz}\right\} = 0, \quad (3.3.2)$$

2) The value function of the control problem in Section 2.2 is given by

$$v(t,x,y) = y^\gamma u\left(t, \frac{x}{y}\right), \forall (t,x,y) \in \mathcal{O},$$

where $u : (0,T] \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is the unique viscosity solution to equation (3.3.1) for $\overline{y} = 1$ with polynomial growth at infinity.

Proof. The proof of Theorem 3.3.1 follows from the one given in [2, Section 2.2]. But here we show first of all that $\overline{u}$ satisfies equation (3.3.1).

- We show that $\overline{u}$ satisfies equation (3.3.1). From Lemma 3.1.1, we have for a fixed $\overline{y} > 0$

$$v(t,x,y) = v\left(t, \frac{y}{\overline{y}}\cdot\overline{y}\frac{x}{y}, \overline{y}\cdot\frac{y}{\overline{y}}\right) = \left(\frac{y}{\overline{y}}\right)^\gamma v\left(t, \overline{y}\frac{x}{y}, \overline{y}\right) = \left(\frac{y}{\overline{y}}\right)^\gamma \overline{u}\left(t, \overline{y}\frac{x}{y}\right) = \left(\frac{y}{\overline{y}}\right)^\gamma \overline{u}(t,z), \quad z = \overline{y}\frac{x}{y}.$$

We use the chain rule ($\frac{\partial\overline{u}}{\partial x} = \frac{\partial\overline{u}}{\partial z}\cdot\frac{\partial z}{\partial x}$) to write the partial derivatives of $\overline{u}$ in terms of the partial derivatives of the value function v, we obtain:

$$v_t = \left(\frac{y}{\overline{y}}\right)^\gamma \overline{u}_t, \qquad v_x = \left(\frac{y}{\overline{y}}\right)^{\gamma-1}\overline{u}_z, \qquad v_{xy} = v_{yx} = (\gamma-1)\left(\frac{y}{\overline{y}}\right)^{\gamma-2}\frac{1}{\overline{y}}\overline{u}_z - \left(\frac{y}{\overline{y}}\right)^{\gamma-2}\frac{x}{y}\overline{u}_{zz},$$
$$v_{xx} = \left(\frac{y}{\overline{y}}\right)^{\gamma-2}\overline{u}_{zz}, \qquad v_y = \gamma\left(\frac{y}{\overline{y}}\right)^{\gamma-1}\frac{1}{\overline{y}}\overline{u} - \left(\frac{y}{\overline{y}}\right)^{\gamma-1}\frac{x}{y}\overline{u}_z,$$
$$v_{yy} = \gamma(\gamma-1)\left(\frac{y}{\overline{y}}\right)^{\gamma-2}\frac{1}{\overline{y}^2}\overline{u} - 2(\gamma-1)\left(\frac{y}{\overline{y}}\right)^{\gamma-2}\frac{1}{\overline{y}}\frac{x}{y}\overline{u}_z + \left(\frac{y}{\overline{y}}\right)^{\gamma-2}\left(\frac{x}{y}\right)^2\overline{u}_{zz}. \quad (3.3.3)$$

At the point $(t,z,\overline{y})$, the partial derivatives (3.3.3) become:

$$v_t = \overline{u}_t, \quad v_x = \overline{u}_z, \quad v_{xx} = \overline{u}_{zz}, \quad v_{xy} = \frac{1}{\overline{y}}[(\gamma-1)\overline{u}_z - z\overline{u}_{zz}],$$
$$v_y = \frac{1}{\overline{y}}[\gamma\overline{u} - z\overline{u}_{zz}], \quad v_{yy} = \frac{1}{\overline{y}^2}[\gamma(\gamma-1)\overline{u} - 2(\gamma-1)\overline{u}_z + z^2\overline{u}_{zz}]. \quad (3.3.4)$$

Substituting (3.3.4) into JHB equation (2.3.9) gives

$$\overline{u}_t + \sup_{\pi\in\mathcal{A}_1} \{(r+\pi_s\sigma\theta)z\overline{u}_z + \overline{y}\overline{u}_z + \mu_C(t)[\gamma\overline{u} - z\overline{u}_{zz}] + \frac{1}{2}(\pi_t\sigma)^2 z^2\overline{u}_{zz} +$$
$$\frac{1}{2}(\sigma_C(t))^2[\gamma(\gamma-1)\overline{u} - 2(\gamma-1)\overline{u}_z + z^2\overline{u}_{zz}] + \rho\sigma\sigma_C(t)\pi z[(\gamma-1)\overline{u}_z - z\overline{u}_{zz}]\} = 0. \quad (3.3.5)$$

Taking out of the supremum terms independent of the control π in (3.3.5) lead to (3.3.1). At the terminal time T we have directly $\overline{u}(T,z) = v(T,z,\overline{y}) = \frac{z^\gamma}{\gamma}$.

• To prove 1) we must prove that the function $\overline{u}$ is both a supersolution and subsolution of (3.3.1). We just show the supersolution property, because the proof of the subsolution property is completely analogous and is already done in [2, Section 2.2].

Define the following operators:

for $(t,x,y)\in\mathcal{O},\ s=v(t,x,y),\ q=v_t(t,x,y),\ p=Dv(t,x,y),\ M=D^2v(t,x,y)$

$$F(t,x,y,s,q,p,M) = -q + \sup_{\pi\in\mathcal{A}_1}\{[-yp_1 - \mu_C(t)yp_2 - (r+\pi_s\sigma\theta)p_1$$
$$-\frac{1}{2}(x\pi_t\sigma)^2M_{11} - \frac{1}{2}(y\sigma_C(t))^2M_{22} + \rho\sigma\sigma_C(t)xy\pi M_{12}\}.$$

For $(t,z)\in[0,T]\times(0,+\infty)$, $\overline{y}$ fixed, $s=\overline{u}(t,z),\ q=\overline{u}_t(t,z),\ p=\frac{\partial\overline{u}(t,z)}{\partial z}\ M=\frac{\partial^2\overline{u}(t,z)}{\partial z^2}$

$$F^{(\overline{y})}(t,z,s,q,p,M) = -q - \overline{y}p - \mu_C(t)[\gamma s - zp] - \frac{1}{2}\mu_C(t)[\gamma(\gamma-1)s - 2(\gamma-1)zp + z^2M]$$
$$-\sup_{\pi\in\mathcal{A}_1}\left\{(\pi\sigma\theta+r)zp + \frac{1}{2}(\pi\sigma)^2z^2M + \rho\sigma_C(t)\sigma\pi(\gamma-1)zp - \rho\sigma_C(t)\sigma\pi z^2M\right\}. \quad (3.3.6)$$

Recall that, by Theorem 3.2.1, v is a solution of equation (3.2.1). In particular, v is a viscosity supersolution, which implies that for each fixed point $(t_0,z_0,\overline{y})\in\mathcal{O}$ and for each test function $\varphi\in C^{1,2}(\mathcal{O})$ such that

$$0 = (v-\varphi)(t_0,z_0,\overline{y}) = \min_{(t,x,y)\in\mathcal{O}}(v-\varphi)(t,x,y), \quad (3.3.7)$$

it holds

$$F(t_0,z_0,\overline{y},v(t_0,z_0,\overline{y}),\varphi_t(t_0,z_0,\overline{y}),D\varphi(t_0,z_0,\overline{y}),D^2\varphi(t_0,z_0,\overline{y}))\geq 0. \quad (3.3.8)$$

We want to show that the function $\overline{u}$ is also a viscosity supersolution of the reduction form (3.3.8), i.e. for every $(t_0,z_0)\in[0,T]\times(0,+\infty)$ and for every $\psi\in C^{1,2}([0,T]\times(0,+\infty))$ such that

$$0 = (\overline{u}-\psi)(t_0,z_0) = \min_{(t,z)\in[0,T]\times(0,+\infty)}(\overline{u}-\psi)(t,z). \quad (3.3.9)$$

It holds

$$F^{(\overline{y})}(t_0,z_0,\overline{u}(t_0,z_0),\psi_t(t_0,z_0),D\psi(t_0,z_0),D^2\psi(t_0,z_0))\geq 0.$$

Let hence $\psi\in C^{1,2}([0,T]\times(0,+\infty))$ such that ψ satisfies (3.3.9), this implies that $\psi(t_0,z_0)=\overline{u}(t_0,z_0)$ and $\psi\leq\overline{u}$ on $[0,T]\times(0,+\infty)$.

Define

$$\varphi(t,x,y) = \left(\frac{y}{\overline{y}}\right)^\gamma \psi\left(t,\overline{y}\frac{x}{y}\right).$$

Then at the point $(t_0, z_0, \overline{y})$,

$$\varphi(t_0, z_0, \overline{y}) = \psi(t_0, z_0) = \overline{u}(t_0, z_0) = v(t_0, z_0, \overline{y})$$

and for every (t, x, y)

$$\varphi(t, x, y) = \left(\frac{y}{\overline{y}}\right)^{\gamma} \psi\left(t, \overline{y}\frac{x}{y}\right) \leq \left(\frac{y}{\overline{y}}\right)^{\gamma} \overline{u}\left(t, \overline{y}\frac{x}{y}\right) = v(t, x, y).$$

Moreover, the function φ is of class $C^{1,2}(\mathcal{O})$ and using the partial derivatives (3.3.4), we deduce the partial derivatives of ψ in terms of the partial derivatives of φ, we obtain at the point $(t_0, z_0, \overline{y})$

$$\varphi_t = \psi_t, \quad \varphi_x = \psi_z, \quad \varphi_{xx} = \psi_{zz}, \quad \varphi_{xy} = \frac{1}{\overline{y}}[(\gamma - 1)\psi_z - z\psi_{zz}],$$
$$\varphi_y = \frac{1}{\overline{y}}[\gamma\psi - z\psi_{zz}], \quad \varphi_{yy} = \frac{1}{\overline{y}^2}[\gamma(\gamma - 1)\psi - 2(\gamma - 1)\psi_z + z^2\psi_{zz}].$$

Then the viscosity supersolution property of the value function yields

$$\begin{aligned}
0 &\leq F(t_0, z_0, \overline{y}, v(t_0, z_0, \overline{y}), \varphi_t(t_0, z_0, \overline{y}), D\varphi(t_0, z_0, \overline{y}), D^2\varphi(t_0, z_0, \overline{y})) \\
&= -\psi_t - \overline{y}\psi_z - \mu_C(t_0)[\gamma\psi - z\psi_z] - \frac{1}{2}\mu_C(t_0)[\gamma(\gamma - 1)\psi - 2(\gamma - 1)z\psi_z + z^2\psi_{zz}] \\
&- \sup_{\pi \in \mathcal{A}_1} \left\{(\pi\sigma\theta + r)z\psi_z + \frac{1}{2}(\pi\sigma)^2 z^2\psi_{zz} + \rho\sigma_C(t_0)\sigma\pi(\gamma - 1)z\psi_z - \rho\sigma_C(t_0)\sigma\pi z^2\psi_{zz}\right\} \\
&= F^{(\overline{y})}(t_0, z_0, \overline{u}(t_0, z_0), \psi_t(t_0, z_0), D\psi(t_0, z_0), D^2\psi(t_0, z_0)).
\end{aligned}$$

This implies that, at the point (t_0, z_0)

$$F^{(\overline{y})}(t_0, z_0, \overline{u}(t_0, z_0), \psi_t(t_0, z_0), D\psi(t_0, z_0), D^2\psi(t_0, z_0)) \geq 0.$$

- The proof of 2) can be found in [2, Theorem 2.4]. □

Up to now, we only know that the value function v is the unique viscosity solution of the HJB equation. Under suitable conditions on the parameters in the dynamics of the endowment process it can be shown that classical solutions to our reduced HJB equation (3.3.2) exist. To obtain regularity of the solution of the HJB equation, we need the following:

Theorem 3.3.2 (Regularity of solution). *Assume that the coefficients $\mu_C, \sigma_C : [0, T] \longrightarrow \mathbb{R}$ are continuously differentiable and that there exists $\varepsilon > 0$ such that $\sigma_C(t) > \varepsilon$ for every $t \in [0, T]$, then the value function $v \in C^{1,2}(\mathcal{O})$.*

Proposition 3.3.1. *The proof Theorem 3.3.2 can be found in [2, Theorem 2.7].*

3.4 The optimal strategy

In order to simplify the optimization problem, we have reduced the HJB equation by one dimension. Due to this, we can then solve the deduced to get the solution of the original one. We can also prove by means of a verification theorem (see [14, Theorem 3.5.2]) that given an optimal strategy for the reduced problem we can derive the optimal strategy for the original one. To prove the optimality of the strategy, we use the regularity of the value function.

Proposition 3.4.1. *Assume that $u \in C^{1,2}$ is a classical solution to equation (3.3.2), then the optimal strategy is Markovian and is given by*

$$\Pi_t = h(t, X_s^{\Pi}, c_t).$$

where

$$h(t,x,y) = -\frac{[\theta - \rho\sigma_C(t)(1-\gamma)]}{\sigma}\frac{y}{x}\frac{u_z\left(t, \frac{x}{y}\right)}{u_{zz}\left(t, \frac{x}{y}\right)} + \frac{\rho\sigma_C(t)}{\sigma}, \text{ if } \Pi_t \text{ belongs to } K \tag{3.4.1}$$

Proof. Let $g : K \longrightarrow \mathbb{R}$ defined by

$$g(p) = (p\sigma\theta + r)\frac{x}{y}u_z\left(t, \frac{x}{y}\right) + \frac{1}{2}(p\sigma)^2\left(\frac{x}{y}\right)^2 u_{zz}\left(t, \frac{x}{y}\right)$$
$$- \rho\sigma_C(t)\sigma p(1-\gamma)\left(\frac{x}{y}\right)u_z\left(t, \frac{x}{y}\right) - \rho\sigma_C(t)\sigma p\left(\frac{x}{y}\right)^2 u_{zz}\left(t, \frac{x}{y}\right).$$

The first derivative of g is given by

$$g'(p) = \sigma\theta\frac{x}{y}u_z\left(t, \frac{x}{y}\right) + p\sigma^2\left(\frac{x}{y}\right)^2 u_{zz}\left(t, \frac{x}{y}\right)$$
$$- \rho\sigma_C(t)\sigma(1-\gamma)\left(\frac{x}{y}\right)u_z\left(t, \frac{x}{y}\right) - \rho\sigma_C(t)\sigma\left(\frac{x}{y}\right)^2 u_{zz}\left(t, \frac{x}{y}\right).$$

Setting $g'(p) = 0$ yields

$$p^* = -\frac{[\theta - \rho\sigma_C(t)(1-\gamma)]}{\sigma}\frac{y}{x}\frac{u_z\left(t, \frac{x}{y}\right)}{u_{zz}\left(t, \frac{x}{y}\right)} + \frac{\rho\sigma_C(t)}{\sigma}.$$

Furthermore,

$$g''(p^*) = \sigma^2\left(\frac{x}{y}\right)^2 u_{zz}\left(t, \frac{x}{y}\right) < 0,$$

since u is strictly concave. Thus g is strictly concave on the closed convex set K. Hence our candidate p^* is a maximizer if Π_t belongs to K.

We conclude that the Markovian strategy is given as

$$\Pi_t = h(t, X_t^{\Pi}, c_t), \tag{3.4.2}$$

where h is given by (3.4.1)

The next step is to prove that the Markovian strategy is optimal, i.e. we want to show that

$$v(t,x,y) = \mathbb{E}\left[U(X_T^{\Pi})\right].$$

Fix $t \in [0,T]$ and let hence $(X_s)_{s\in[t,T]}$ be the solution to

$$dX_s = [X_s(r + \Pi_s\sigma\theta) + c_s]ds + X_s\Pi_s\sigma dW_s^1, \quad \forall s \in (t, T], \; X_t = x$$
$$dc_s = \mu_C(s)c_s ds + \sigma_C(s)c_s dW^C, \quad \forall s \in (t, T] \; c_t = y,$$

with Π as in equation (3.4.2).

Define

$$Z_s := \frac{X_s}{c_s}, \quad U_s = u(s, Z_s) \text{ and } V_s = c_s^\gamma U_s.$$

Since Z is a well-defined semimartingale, and $u(s, .)$ is a concave function, the process U_s is also a well-defined semimartingale. This is a consequence of an application of Itô formula, which yields:

- For the process Z:

$$\begin{aligned} dZ_s = Z_s\{&-\mu_C(s) + \sigma_C^2(s) - \Pi_s\rho\sigma\sigma_C(s) + \Pi_s\theta\sigma + r\}ds + ds+ \\ &Z_s\{\Pi_s\sigma dW_s^1 - \sigma_C(s)dW_s^C\}. \end{aligned} \tag{3.4.3}$$

Indeed, the partial integration gives

$$dZ_s = X_s d\frac{1}{c_s} + \frac{1}{c_s}dX_s + d\left\langle X, \frac{1}{c}\right\rangle_s \tag{3.4.4}$$

Let $f(x) = \frac{1}{x}$. Then

$$f'(x) = -\frac{1}{x^2} \text{ and } f''(x) = \frac{2}{x^3}.$$

We apply the Itô formula to the function $f(c_s) = \frac{1}{c_s}$ to obtain

$$\begin{aligned} d\frac{1}{c_s} &= -\frac{1}{c_s^2}dc_s + \frac{1}{2}\left(\frac{2}{c_s^3}\right)d\langle c\rangle_s \\ &= -\frac{1}{c^2}\left\{\mu_C(s)c_s ds + \sigma_C(s)c_s dW^C\right\} + \frac{c_s^2\sigma_C^2(s)}{c_s^3}ds \\ &= \frac{1}{c_s}(\sigma_C^2(s) - \mu_C(s))ds - \frac{\sigma_C(s)}{c_s}dW_s^C. \end{aligned} \tag{3.4.5}$$

Since W^C is correlated with W^1, with correlation coefficient ρ, we then have

$$d\left\langle X, \frac{1}{c}\right\rangle_s = -\frac{X_s}{c_s}\rho\sigma\sigma_C(s)\Pi_s ds. \tag{3.4.6}$$

Substituting (3.4.5) and (3.4.6) into (3.4.4), to get

$$\begin{aligned} dZ_s = &\frac{X_s}{c_s}[(r + \Pi_s\sigma\theta)ds + \Pi_s\sigma dW_s^1] + ds+ \\ &X_s\left\{\frac{1}{c_s}(\sigma_C^2(s) - \mu_C(s))ds - \frac{\sigma_C(s)}{c_s}dW_s^C\right\} - \frac{X_s}{c_s}\rho\sigma\sigma_C(s)\Pi_s ds. \end{aligned} \tag{3.4.7}$$

Substituting $Z_s := \frac{X_s}{c_s}$, to obtain (3.4.3).

- For the process $U_s = u(s, Z_s)$, under the assumption that $u \in C^{1,2}$,

$$\begin{aligned} dU_s = \{&u_s(s,Z_s) + u_z(s,Z_s) + Z_s u_z(s,Z_s)[-\mu_C(s) + \sigma_C^2(s) - \Pi_s\rho\sigma\sigma_C(s) + \Pi_s\theta\sigma + r] \\ &+ \frac{1}{2}(\Pi_s\sigma)^2 Z^2 u_{zz}(s,Z_s) + \frac{1}{2}\sigma_C^2(s)Z_s^2 u_{zz}(s,Z_s) - \rho\sigma\sigma_C(s)\Pi_s Z_s^2 u_{zz}(s,Z_s)\}ds \\ &+ Z_s u_z(s,Z_s)\{\Pi_s\sigma dW_s^1 - \sigma_C(s)dW_s^C\}. \end{aligned} \tag{3.4.8}$$

Indeed, the Itô formula for time-dependent functions applying to the function $U_s = u(s, Z_s)$, gives

$$dU_s = u_s(s,Z_s)ds + u_z(s,Z_s)dZ_s + \frac{1}{2}u_{zz}(s,Z_s)d\langle Z\rangle_s. \tag{3.4.9}$$

Notice that Z is driven by two correlated Wiener processes W^1 and W^C.

Then

$$\begin{aligned} d\langle Z\rangle_s &= d\langle Z_s\Pi_s\sigma W^1 - Z_s\sigma_C(s)W^C\rangle_s \\ &= d\langle Z_s\Pi_s\sigma W^1\rangle_s + d\langle Z_s\sigma_C(s)W^C\rangle_s - 2d\langle Z_s\Pi_s\sigma W^1, Z_s\sigma_C(s)W^C\rangle_s \\ &= \{(\Pi_s\sigma)^2 Z^2 + \sigma_C^2(s)Z_s^2 - 2\rho\sigma\sigma_C(s)\Pi_s Z_s^2\}ds. \end{aligned} \tag{3.4.10}$$

Substituting (3.4.3) and (3.4.10) into (3.4.9), we get (3.4.8).

- Finally, for the function $V_s = c_s^\gamma u(s, Z_s)$ we get again by the partial integration formula

$$\begin{aligned} dV_s = c_s^\gamma\{&u_s(s,Z_s) + u_z(s,Z_s) + Z_s u_z(s,Z_s)[-\mu_C(s) + (1-\gamma)\sigma_C^2(s) - (1-\gamma)\Pi_s\rho\sigma\sigma_C(s) + \Pi_s\theta\sigma + r] \\ &+ \frac{1}{2}(\Pi_s\sigma)^2 Z^2 u_{zz}(s,Z_s) + \frac{1}{2}\sigma_C^2(s)Z_s^2 u_{zz}(s,Z_s) - \rho\sigma\sigma_C(s)\Pi_s Z_s^2 u_{zz}(s,Z_s) \\ &+ \gamma u(s,Z_s)\left(\mu_C(s) + \frac{\gamma-1}{2}\sigma_C^2(s)\right)\}ds \\ &+ c_s^\gamma\{Z_s u_z(s,Z_s)\Pi_s\sigma dW_s^1 + [\gamma\sigma_C(s)u(s,Z_s) - Z_s u_z(s,Z_s)\sigma_C(s)]dW_s^C\}. \end{aligned} \tag{3.4.11}$$

Indeed, the partial integration applying to $V_s = c_s^\gamma U_s$ gives

$$dV_s = c_s^\gamma dU_s + u(s,Z_s)dc_s^\gamma + d\langle U_s, c_s^\gamma\rangle_s. \tag{3.4.12}$$

Let $g(x) = x^\gamma$.

Then

$$g'(x) = \gamma x^{\gamma-1} \text{ and } g''(x) = \gamma(\gamma-1)x^{\gamma-2}.$$

We get by the one-dimensional Itô formula

$$\begin{aligned} dc_s^\gamma &= \gamma c_s^{\gamma-1}dc_s + \frac{1}{2}\gamma(\gamma-1)c_s^{\gamma-2}d\langle c\rangle_s \\ &= \gamma c_s^{\gamma-1}[\mu_C(s)c_s ds + \sigma_C(s)c_s dW^C] + \frac{1}{2}\gamma(\gamma-1)c_s^\gamma\sigma_C^2(s)ds \\ &= \gamma c_s^\gamma\left(\mu_C(s) + \frac{\gamma-1}{2}\sigma_C^2(s)\right)ds + \gamma c_s^\gamma\sigma_C(s)dW_s^C. \end{aligned} \tag{3.4.13}$$

Due to the fact that W^1 and W^C are correlated,

$$\begin{aligned} d\langle U_s, c_s^\gamma\rangle_s &= d\langle Z_s u_z(s,Z_s)\Pi_s\sigma dW_s^1 - Z_s u_z(s,Z_s)\sigma_C(s)dW_s^C, \gamma c_s^\gamma\sigma_C(s)W^C\rangle_s \\ &= (Z_s u_z(s,Z_s)\gamma c_s^\gamma\rho\sigma\sigma_C(s)\Pi_s - Z_s u_z(s,Z_s)\gamma c_s^\gamma\sigma_C(s)^2)ds. \end{aligned} \tag{3.4.14}$$

Substituting (3.4.13) and (3.4.14) into (3.4.12), we get (3.4.11).

By definition

$$V_s = c_s^\gamma u(s,Z_s) = c_s^\gamma v\left(s, \frac{X_s}{c_s}, 1\right) = c_s^\gamma v\left(s, \frac{X_s}{c_s}, \frac{1}{c_s}\cdot c_s\right) = c_s^\gamma \frac{1}{c_s^\gamma} v(s, X_s, c_s) \text{ by the homogeneity of } v$$

then

$$V_s = v(s, X_s, c_s).$$

Since u satisfies the HJB equation (3.3.2) and for a strategy $\Pi = \Pi_s^*$ the supremum is attained. The stochatic differential dV_s in (3.4.11) can be written as

$$dV_s = c_s^\gamma\{0\cdot ds + Z_s u_z(s,Z_s)\Pi_s\sigma dW_s^1 + [\gamma\sigma_C(s)u(s,Z_s) - Z_s u_z(s,Z_s)\sigma_C(s)]dW_s^C\}. \tag{3.4.15}$$

Integrating both sides of (3.4.15) yields

$$V_T - V_t = \int_t^T c_s^\gamma\{Z_s u_z(s,Z_s)\Pi_s\sigma dW_s^1 + [\gamma\sigma_C(s)u(s,Z_s) - Z_s u_z(s,Z_s)\sigma_C(s)]dW_s^C\}.$$

Using the terminal condition and the fact that the process starts at time t from (x,y), we have

$$V_T = U(X_T^{\pi^*}) = v(t,x,y) + \underbrace{\int_t^T c_s^\gamma Z_s u_z(s,Z_s)\Pi_s\sigma dW_s^1 + c_s^\gamma[\gamma\sigma_C(s)u(s,Z_s) - Z_s u_z(s,Z_s)\sigma_C(s)]dW_s^C}_{M_T}.$$

Using the fact that M_T is a local martingale ($\mathbb{E}(M_T) = 0$), we have

$$\mathbb{E}\left[U(X_T^{\Pi^*})\right] = v(t,x,y) + 0.$$

Hence

$$v(t,x,y) = \mathbb{E}\left[U(X_T^{\Pi})\right],$$

which shows that the Markovian control Π is optimal.

□

4. Numerical scheme for the HJB equation

Introduction

This chapter introduces the practical aspects of designing finite difference schemes for HJB equation (3.3.2). The approach is based on the very powerful and framework developed in [6, Chapter 10]. Peter Forsyth proved in [9] using the same approach as in [16] that using viscosity solutions techniques, any consistent, monotone and stable approximation scheme is convergent. The key property here is the monotonicity which guarantees that the scheme satisfies the same ellipticity condition as the HJB operator (see [17]). It is worth insisting on the fact that if the scheme is not monotone, it may fail to converge to the correct solution. One of the merits of finite difference schemes is that they are simple and easy to implement. They can also be combined with Monte Carlo methods to solve nonlinear parabolic PDEs (see [7]). Our primary sources for this chapter are [6, 7, 9, 16].

For the numerical solution of the HJB equation we have to truncate the unbounded domain $[0, +\infty)$ for z to the bounded domain $[0, \overline{z}]$ and to define boundary conditions for $z = \overline{z}$. We begin by the asymptotic behaviour of u for $z \to 0$ and $z \to \infty$.

4.1 Asymptotic behaviour

In this section, we examine the asymptotics of the value function u and the optimal strategy π for $z \to 0$ and $z \to \infty$.

4.1.1 Case $z \to 0$. To study the behaviour of u as $z \to 0$, we apply the Fichera theory developed in [1] to the HJB equation (3.3.2).

Proposition 4.1.1. *Let u be the classical solution of the HJB equation (3.3.2), then no boundary condition is required at $z \to 0$.*

Proof. Let a, b and c denote the coefficients of u_{zz}, u_z and u respectively by

$$a(t, z, p) := \frac{1}{2}(p\sigma)^2 z^2 + \frac{1}{2}\sigma_C^2(t) z^2 - \rho\sigma\sigma_C(t) p z^2, \tag{4.1.1}$$

$$b(t, z, p) := 1 - \mu_C(t) z + (1-\gamma)\sigma_C^2(t) z + (\gamma - 1) p\rho\sigma\sigma_C(t) z + (p\theta\sigma + r) z, \tag{4.1.2}$$

$$c(t, z, p) := -\gamma \left(\mu_C(t) + \frac{\gamma - 1}{2}\sigma_C^2(t) \right), \tag{4.1.3}$$

and denote by $\mathcal{L}^p$ the operator

$$\mathcal{L}^p u = a(t, z, p) u_{zz} + b(t, z, p) u_z - c(t, z, p) u.$$

Then equation (3.3.2) reads

$$-u_t = \sup_{p \in K} \mathcal{L}^p u,$$

We define the Fichera function by

$$f(z) = b(t, z, p) - \frac{\partial a(t, z, p)}{\partial z}. \tag{4.1.4}$$

so that for every $t \in [0,T]$ and for every $p \in K$

$$\lim_{z\to 0} f(z) = 1 > 0,$$

which implies that no boundary condition is required at $\{z = 0\}$. □

We have for $z \to 0$ a degenerated HJB equation

$$u_t = -u_z - \gamma\left(\mu_C(s) + \frac{\gamma-1}{2}\sigma_C^2(s)\right)u, \tag{4.1.5}$$

which does not contain the strategy π. So we cannot obtain the optimal strategy π^* by solving the pointwise optimization problem $(\sup_{p\in K} \mathcal{L}^p u)$ in the HJB equation. $z = \frac{x}{y} = 0$ means that we have infinite endowment at time t. So the investor can invest arbitrarily high capital (and not only his wealth x) to the risky asset. Depending on the correlation ρ this leads to $\Pi = \pm\infty$.

4.1.2 Case $z \to \infty$. When z becomes large ($z = \bar{z}$ for some large $\bar{z}$), we can get some asymptotics for the value function u and the optimal strategy Π, as stated in the following.

Proposition 4.1.2. *Let u be the classical solution of the HJB equation (3.3.2), then it holds that*

(i) for $z \to +\infty$,

$$u(t,z) \simeq \frac{z^\gamma}{\gamma}\exp\left\{\gamma\left(r + \frac{\theta^2}{2(1-\gamma)}\right)(T-t)\right\}. \tag{4.1.6}$$

(ii) The optimal strategy defined as in (3.4.1) is such that, for z

$$\Pi = \Pi(t,z) \simeq \frac{\theta}{\sigma(1-\gamma)}, \tag{4.1.7}$$

if it is in $\mathcal{A}_1$, i.e. it approaches to the Merton ratio.

Proof. The proof of proposition 4.1.2 can be found in [2, Proposition 3.4]. □

This leads us to the Dirichlet boundary condition given by equation (4.1.6) for the HJB equation (3.3.2) on the truncated domain $[0,T]\times[0,\bar{z}]$, for some large $\bar{z}$.

4.2 Problem reformulation

Let $\tau = T - t$ be the time to maturity. Define

$$\tilde{\mu}_C(\tau) = \mu_C(T-\tau),\ \ \tilde{\sigma}_C(\tau) = \sigma_C(T-\tau) \text{ and } \ \tilde{u}(\tau,z) = u(T-\tau,z) = u(t,z).$$

Then equation (3.3.2) becomes

$$\begin{aligned}\tilde{u}_\tau = \tilde{u}_z + \tilde{\mu}_C(\tau)[\gamma\tilde{u} - z\tilde{u}_z] + \frac{1}{2}\tilde{\sigma}_C^2(\tau)[\gamma(\gamma-1)\tilde{u} - 2(\gamma-1)z\tilde{u}_z + z^2\tilde{u}_{zz}] +\\ \sup_{p\in K}\left\{(p\sigma\theta + r)z\tilde{u}_z + \frac{1}{2}(p\sigma)^2z^2\tilde{u}_{zz} + \rho\tilde{\sigma}_C(\tau)\sigma p(\gamma-1)z\tilde{u}_z - \rho\tilde{\sigma}_C(\tau)\sigma p z^2\tilde{u}_{zz}\right\} = 0.\end{aligned} \tag{4.2.1}$$

Using the above definition, the optimal strategy (3.4.2) becomes

$$p^* = p^*(t,z) = -\frac{[\theta - \rho\tilde{\sigma}_C(\tau)(1-\gamma)]}{\sigma}\frac{y}{x}\frac{\tilde{u}_z\left(\tau,\frac{x}{y}\right)}{\tilde{u}_{zz}\left(\tau,\frac{x}{y}\right)} + \frac{\rho\tilde{\sigma}_C(\tau)}{\sigma}. \tag{4.2.2}$$

Substituting (4.2.2) into (4.2.1) yields

$$\tilde{u}_\tau = a(\tau,z)\tilde{u}_{zz} + b(\tau,z)\tilde{u}_z - c(\tau,z)\tilde{u} + d(\tau,z)\frac{\tilde{u}_z^2}{\tilde{u}_{zz}}, \tag{4.2.3}$$

where

$$a(\tau,z) := \frac{1}{2}(1-\rho^2)z^2\tilde{\sigma}_C^2(\tau), \tag{4.2.4}$$

$$b(\tau,z) := 1 - z(\tilde{\mu}_C(\tau) - r) + z\tilde{\sigma}_C(\tau)[\rho\theta - (1-\rho^2)(\gamma-1)\tilde{\sigma}_C(\tau)], \tag{4.2.5}$$

$$c(\tau,z) := -\gamma\left(\tilde{\mu}_C(\tau) + \frac{\gamma-1}{2}\tilde{\sigma}_C^2(\tau)\right), \tag{4.2.6}$$

$$d(\tau,z) := -\frac{1}{2}[\theta + (\gamma-1)\rho\tilde{\sigma}_C(\tau)]^2. \tag{4.2.7}$$

4.2.1 Condition. The coefficients of the equation (4.2.3) satisfy the following conditions:

1) $a(\tau,z) \geq 0$ and $d(\tau,z) \leq 0$.

2) For the coefficient $b = b(\tau,z)$ we write.

$$b = b^+ - b^-,$$

where

$$b^+ = |b|\mathbf{1}_{\{b\geq 0\}} = \frac{b(\tau,z) + |b(\tau,z)|}{2},$$
$$b^- = |b|\mathbf{1}_{\{b<0\}} = -\frac{b(\tau,z) - |b(\tau,z)|}{2}$$

Note that $b^+ = b$ and $b^- = 0$ for $b > 0$, and $b^+ = 0$ and $b^- = b$ for $b < 0$.

3) For the coefficient $c(\tau,z)$, we have

(i) for $0 < \gamma < 1$

$$c(\tau,z) \geq 0 \quad \text{if} \quad \tilde{\mu}_C(\tau) \leq \overline{\mu}_C,$$
$$c(\tau,z) < 0 \quad \text{if} \quad \tilde{\mu}_C(\tau) > \overline{\mu}_C.$$

(ii) For $\gamma < 0$

$$c(\tau,z) \geq 0 \quad \text{if} \quad \tilde{\mu}_C(\tau) \geq \overline{\mu}_C,$$
$$c(\tau,z) < 0 \quad \text{if} \quad \tilde{\mu}_C(\tau) < \overline{\mu}_C.$$

with $\overline{\mu}_C = \frac{(1-\gamma)\tilde{\sigma}_C^2(\tau)}{2}$

4.3 Finite difference scheme

Let $0 = \tau_0 < \tau_1 < \cdots < \tau_{N_\tau} = T$ be an equidistant partition of the interval $[0, T]$, i.e. $\tau_n = n\Delta\tau,\ n = 0, \cdots,\ N_\tau$ where $\Delta\tau = \frac{T}{N_\tau},\ N_\tau \geq 1$ is the mesh size in the τ-direction and let $\tau_{n+\frac{1}{2}} = \frac{\tau_{n+1}+\tau_n}{2}$. For the z-direction we consider an initial equidistant grid of $[0, 1]$ and let

$$z = \overline{z}\frac{\tan(\frac{k\pi}{2}x)}{\tan(\frac{k\pi}{2})}, \quad \text{for some} \quad k \in (0, 1)$$

be the function which transforms the initial equidistant grid of $[0, 1]$ to a non-equidistant grid of $[0, \overline{z}]$. The idea of the transformation is to have many grid point where u is strongly varying and less grid point where u is slowly varying. In fact in order to ensure the *stability* of the scheme if Δz_j is too small then we have to use too small time steps $\Delta\tau$ or too large N_τ where $\Delta z_j = z_j - z_{j-1},\quad j = 1, \cdots, N_z$ be the mesh sizes in the z direction. In the case where the coefficients are not constant, to get a higher approximation order (quadratic), we evaluate the coefficients at an appropriate point between τ_n and τ_{n+1} such as $\tau_{n+\frac{1}{2}}$. Set for $j = 1, \quad \cdots \quad N_z - 1$

$$a_j^{n+\frac{1}{2}} = a(\tau_{n+\frac{1}{2}}, z_j),\ b_j^{n+\frac{1}{2}} = b(\tau_{n+\frac{1}{2}}, z_j),\ c_j^{n+\frac{1}{2}} = c(\tau_{n+\frac{1}{2}}, z_j) \text{ and } d_j^{n+\frac{1}{2}} = d(\tau_{n+\frac{1}{2}}, z_j).$$

Let $\tilde{u}_j^n$ be the discrete approximation of $\tilde{u}(\tau, z)$ at node (τ_n, z_j) and set $\tilde{u}^n = [\tilde{u}_0^n, \cdots, \tilde{u}_{N_z}]^T$.

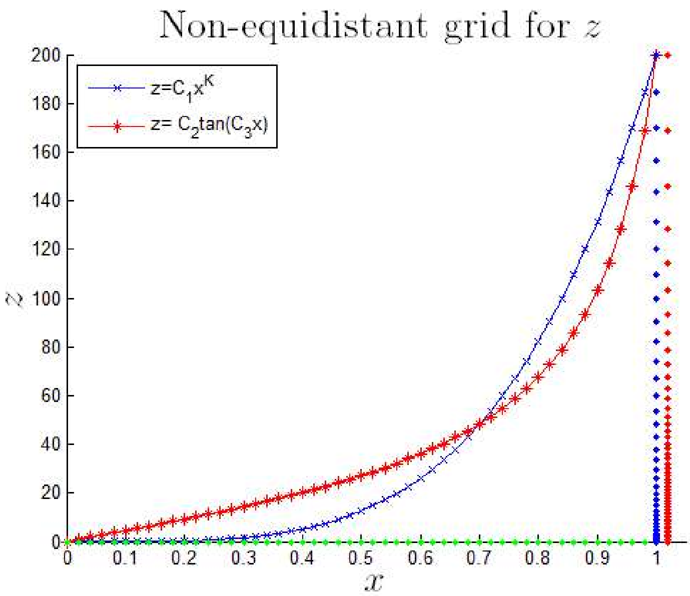


Figure 4.1: Non-equidistant grid for z.

4.3.1 Treatment of the linear part. For improving the stability of finite difference schemes the idea is that depending on the sign of the coefficient $b_j^{n+\frac{1}{2}}$ in front of first-order derivative u_z (convection terms) one uses forward or backward differences:

For $b_j^{n+\frac{1}{2}} > 0$: one uses implicit forward difference $u_z \simeq \frac{\tilde{u}_{j+1}^{n+1} - \tilde{u}_j^{n+1}}{z_{j+1} - z_j}$.

For $b_j^{n+\frac{1}{2}} < 0$: one uses implicit backward difference $u_z \simeq \frac{\tilde{u}_j^{n+1} - \tilde{u}_{j-1}^{n+1}}{z_j - z_{j-1}}$.

For the diffusion therm we use the implicit second order difference to obtain

$$\tilde{u}_{zz} \simeq \left(\frac{\tilde{u}_{j+1}^{n+1} - \tilde{u}_j^{n+1}}{z_{j+1} - z_j} - \frac{\tilde{u}_j^{n+1} - \tilde{u}_{j-1}^{n+1}}{z_j - z_{j-1}} \right) \left(\frac{2}{z_{j+1} - z_{j-1}} \right).$$

Using condition 4.2.1, we discretize

$$c(\tau, z)\tilde{u} \simeq c_j^{n+\frac{1}{2}} \mathbf{1}_{\{c_j^{n+\frac{1}{2}} \geq 0\}} \tilde{u}_j^{n+1} + c_j^{n+\frac{1}{2}} \mathbf{1}_{\{c_j^{n+\frac{1}{2}} < 0\}} \tilde{u}_j^n.$$

4.3.2 Treatment of the non-linear part. For the non-linear term we apply the explicit forward difference and the second order difference.

$$\frac{\tilde{u}_z^2}{\tilde{u}_{zz}} \simeq \left(\frac{\tilde{u}_{j+1}^n - \tilde{u}_j^n}{z_{j+1} - z_j} \right)^2 \left(\frac{\tilde{u}_{j+1}^n - \tilde{u}_j^n}{z_{j+1} - z_j} - \frac{\tilde{u}_j^n - \tilde{u}_{j-1}^n}{z_j - z_{j-1}} \right)^{-1} \left(\frac{z_{j+1} - z_{j-1}}{2} \right).$$

4.3.3 Global scheme. On the basis of previous considerations we can establish a global scheme for the discretization of the HJB equation (4.2.3), we obtain the following result:

$$-B_j^{n+\frac{1}{2}} \tilde{u}_{j-1}^{n+1} + C_j^{n+\frac{1}{2}} \tilde{u}_j^{n+1} - A_j^{n+\frac{1}{2}} \tilde{u}_{j+1}^{n+1} = F_j^n, \quad j = 1, \cdots, N_z - 1 \tag{4.3.1}$$

where the discrete equation coefficients are given as

$$\begin{aligned} A_j^{n+\frac{1}{2}} &= \frac{2a_j^{n+\frac{1}{2}}}{(z_{j+1} - z_j)(z_{j+i} - z_{j-1})} + \frac{b_j^{n+\frac{1}{2}} + |b_j^{n+\frac{1}{2}}|}{2(z_{j+1} - z_j)}, \\ B_j^{n+\frac{1}{2}} &= \frac{2a_j^{n+\frac{1}{2}}}{(z_j - z_{j-1})(z_{j+i} - z_{j-1})} - \frac{(b_j^{n+\frac{1}{2}} - |b_j^{n+\frac{1}{2}}|)}{2(z_j - z_{j-1})}, \\ C_j^{n+\frac{1}{2}} &= \frac{1}{\Delta\tau} + c_j^{n+\frac{1}{2}} \mathbf{1}_{\{c_j^{n+\frac{1}{2}} \geq 0\}} + A_j^{n+\frac{1}{2}} + B_j^{n+\frac{1}{2}}. \end{aligned}$$

And the right hand side is given as

$$F_j^n = \tilde{u}_j^n \left(\frac{1}{\Delta\tau} - c_j^{n+\frac{1}{2}} \mathbf{1}_{\{c_j^{n+\frac{1}{2}} < 0\}} \right) + d_j^{n+\frac{1}{2}} \left(\frac{\tilde{u}_{j+1}^n - \tilde{u}_j^n}{z_{j+1} - z_j} \right)^2 \left(\frac{\tilde{u}_{j+1}^n - \tilde{u}_j^n}{z_{j+1} - z_j} - \frac{\tilde{u}_j^n - \tilde{u}_{j-1}^n}{z_j - z_{j-1}} \right)^{-1} \left(\frac{z_{j+1} - z_{j-1}}{2} \right).$$

The optimal strategy is given as

$$\pi_j^n = -\frac{\lambda_1}{z_j} \left(\frac{\tilde{u}_{j+1}^n - \tilde{u}_j^n}{z_{j+1} - z_j} \right) \left(\frac{\tilde{u}_{j+1}^n - \tilde{u}_j^n}{z_{j+1} - z_j} - \frac{\tilde{u}_j^n - \tilde{u}_{j-1}^n}{z_j - z_{j-1}} \right)^{-1} \left(\frac{z_{j+1} - z_{j-1}}{2} \right) + \lambda_2, \quad j = 1, \cdots, N_z - 1$$

where

$$\lambda_1 = \frac{\theta + (\gamma - 1)\rho\tilde{\sigma}_C(t_{n+\frac{1}{2}})}{\rho} \quad \text{and} \quad \lambda_2 = \frac{\rho\tilde{\sigma}_C(t_{n+\frac{1}{2}})}{\sigma}.$$

Condition 4.3.1. *To ensure the convergence of the scheme, the coefficients must satisfied the* positive coefficient condition

$$A_j^{n+\frac{1}{2}} > 0, \quad B_j^{n+\frac{1}{2}} > 0 \ \textit{ and } \ C_j^{n+\frac{1}{2}} - A_j^{n+\frac{1}{2}} - B_j^{n+\frac{1}{2}} > 0, \quad \forall n \geq 0, \quad j = 1, \cdots, N_z - 1. \tag{4.3.2}$$

4.3.4 Boundary conditions. For $z = 0$ we have a degenerated PDE

$$\tilde{u}_\tau = \tilde{u}_z - c(\tau, z)\tilde{u}.$$

Using the implicit forward differences we get

$$\frac{\tilde{u}_0^{n+1} - \tilde{u}_0^n}{\Delta\tau} = \frac{\tilde{u}_1^{n+1} - \tilde{u}_0^{n+1}}{z_1 - z_0} - c_0^{n+\frac{1}{2}} \mathbf{1}_{\{c_0^{n+\frac{1}{2}} \geq 0\}} \tilde{u}_0^{n+1} - c_0^{n+\frac{1}{2}} \mathbf{1}_{\{c_0^{n+\frac{1}{2}} < 0\}} \tilde{u}_0^n,$$

which can be written as

$$C_0^{n+\frac{1}{2}} \tilde{u}_0^{n+1} - A_0^{n+\frac{1}{2}} \tilde{u}_1^{n+1} = F_0^n,$$

where

$$A_0^{n+\frac{1}{2}} = \frac{1}{z_1 - z_0} > 0,$$

$$C_0^{n+\frac{1}{2}} = \frac{1}{\Delta\tau} + c_0^{n+\frac{1}{2}} \mathbf{1}_{\{c_0^{n+\frac{1}{2}} \geq 0\}} + A_0^{n+\frac{1}{2}} > 0,$$

$$F_0^n = \tilde{u}_0^n \left(\frac{1}{\Delta\tau} - c_0^{n+\frac{1}{2}} \mathbf{1}_{\{c_0^{n+\frac{1}{2}} < 0\}} \right).$$

For $z = \overline{z}$ $(j = N)$, we impose the Dirichlet boundary condition (see Proposition 4.1.2)

$$F_{N_z}^n \simeq \frac{z_{N_z}^\gamma}{\gamma} \exp\left\{ \gamma \left(r + \frac{\theta^2}{2(1-\gamma)} \right) \tau_n \right\}, \quad \forall n \geq 0 \text{ and } \pi_{N_z}^n \simeq -\frac{\theta}{\sigma(\gamma - 1)}.$$

4.3.5 Matrix form of the discrete equations. It will be convenient to use matrix notation for equations (4.3.1), coupled with boundary conditions 4.3.4. Let $F^n = [F_0^n, \cdots, F_{N_z}^n]^T$. Then we can write the finite difference scheme (4.3.1) as

$$A^{n+1} \tilde{u}^{n+1} = F^n, \qquad \forall n \geq 0 \tag{4.3.3}$$

where

$$A^{n+1} = \begin{pmatrix} C_0^{n+\frac{1}{2}} & -A_0^{n+\frac{1}{2}} & & 0 \\ -B_1^{n+\frac{1}{2}} & C_1^{n+\frac{1}{2}} & -A_1^{n+\frac{1}{2}} & \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & -A_{N_z-1}^{n+\frac{1}{2}} \\ 0 & & -B_{N_z}^{n+\frac{1}{2}} & C_{N_z}^{n+\frac{1}{2}} \end{pmatrix}.$$

4.4 Numerical results

In this section we used Matlab version 7.10 (R2010a) 32 bits to solve the HJB equation (3.3.2) applying finite difference scheme we have developed in Section 4.3. We take μ_C and σ_C constant and we specify the model for $T = 1, 5$ or 20 years, by taking the parameters from [2] given as in Table 4.1 and using the non-equidistant grid given in the Figure 4.1 in Section 4.3. In order to ensure the stability we have to use a small volatility $\sigma = 0.13$ for $\gamma = -1$ and the higher volatility $\sigma = 0.2$ for $\gamma = 0.5$.

Parameter	σ	γ	μ	σ_C	μ_C	ρ	r	N_z	N_τ	k	σ	γ
Value	0.13	-1	0.04	0.2	0.02	-0.5	0	400	175200	0.85	0.2	0.5

Table 4.1: Parameters of the control problem.

For $T = 20$ years we obtain that the optimal proportion as a function of the ratio wealth to income $z = \frac{x}{y}$ and of the time to maturity $T - t$ to be invested in the stock is given as in Figure 4.2a and 4.2b respectively for $\gamma = -1$ and $\gamma = 0.5$.

Figures 4.2c and 4.2d give the value function for $T = 20$ as a function of the ratio wealth to income $z = \frac{x}{y}$ and of the time to maturity $T - t$ for $\gamma = -1$ and $\gamma = 0.5$ respectively. If we look at both figures we will see that the value function u is increasing and concave in z, and increasing in $T - t$. For $\gamma = -1$, u has a singularity at $T - t = 0$ and is bounded from above by 0 and for $\gamma = 0.5$, u is bounded from below by 0. The case $T = 1$ and $T = 5$ are quite similar.

If we look at the three Figures we will see that for $\gamma = 0.5$ independent of the value of T, the optimal strategy in bounded from below by the Merton ratio and converges towards it as the initial wealth significantly exceeds the initial endowment (as $z \to +\infty$), or as $T - t \to 0$. This is actually perfectly intuitive, since the amount available to the investor is always higher compared to the case without endowments. This actually confirms the asymptotic behaviour that we have studied in Section 4.1. We can also observe that for $\rho = -0.5$ when the initial wealth tends to zero ($z \to 0$), the optimal strategy to be invest in the stock is $\pi^* = +\infty$ and confirms the result in Section 4.1.

For $T = 20$ the results of a sensitivity study with respect to the correlation parameter ρ are given on Figure 4.2g and 4.2h respectively for $\gamma = -1$ and $\gamma = 0.5$. There are indeed two main factors driving the choice of the investor:

(i) the presence of a strictly positive random endowment, which allows for higher investments in the risky asset,

(ii) and the correlation between the random endowment and the risky asset, which allows to hedge partially away risk from the random future endowment by the choice of the investment strategy.

If we look at Figure 4.2g and 4.2h we will observe that:

a) For $\rho = 0$ (case where the risky asset and the random endowment are uncorrelated), no hedging is possible. Future endowments allow to take higher risks and imply therefore an higher investment in the risky asset.

b) For $\rho > 0$ (In the case of extremely positive correlation such as $\rho = 0.95$), the proportion invested in the risky asset is below the Merton ratio for $\gamma = -1$ and there is a possibility to hedge away the risk in the endowment by *short selling* the stock. The case $\rho = 0.5$ results in smaller investments in the stock compared to the case $\rho = 0$.

c) For $\rho < 0$ (In the case of negative correlation), it is possible to hedge away the risk in the endowment by investing *long* in the stock. This results in higher investments in the stock, compared to the case $\rho = 0$.

   The cases $T = 5$ and $T = 1$ are similar to the case $t = 20$.

Since we are not able to reproduce exactly the results from the paper of Chen et al. we have to use another technique to solve the HJB equation (3.3.2) and compare the results with the previous one.

4.5 Policy Improvement Algorithm

Consider the stochastic process Z with the dynamics given by equation (3.4.3). Consider the optimisation problem:

$$\text{Find} \quad u(t,z) = \sup_{\pi \in \mathcal{A}_1} J(t,z,\pi),$$

where $J^{\pi} = J(t,z,\pi) = \mathbb{E}[U(Z_T^{\pi})]$ is the performance criterion and u is the value function given by (3.3.2). For a given Markovian strategy $\tilde{\pi}_{\tau} = p(\tau,z)$ it can be derived that $J^{\tilde{\pi}}$ satisfies the following linear PDE for $H = H(\tau,z)$

$$H_{\tau} = a(\tau,z,p)H_{zz} + b(\tau,z,p)H_z - c(\tau,z,p)H,$$

with initial condition.

$$H(0,z) = U(z) = \frac{z^{\gamma}}{\gamma},$$

where

$$a(\tau,z,p) := \frac{1}{2}(p\sigma)^2 z^2 + \frac{1}{2}\tilde{\sigma}_C^2(\tau)z^2 - \rho\sigma\tilde{\sigma}_C(\tau)pz^2, \tag{4.5.1}$$

$$b(\tau,z,p) := 1 - \tilde{\mu}_C(\tau)z + (1-\gamma)\tilde{\sigma}_C^2(\tau)z + (\gamma-1)\rho\sigma\tilde{\sigma}_C(\tau)pz + (p\theta\sigma + r)z, \tag{4.5.2}$$

$$c(\tau,z,p) := -\gamma\left(\tilde{\mu}_C(\tau) + \frac{\gamma-1}{2}\tilde{\sigma}_C^2(\tau)\right), \tag{4.5.3}$$

with

$$\tilde{\mu}_C(\tau) = \mu_C(T-\tau), \;\; \tilde{\sigma}_C(\tau) = \sigma_C(T-\tau) \;\; \text{and } p = p(\tau,z).$$

4.5.1 Algorithm.

i) Find an initial guess π^0 of the optimal control, set $k=0, \quad \varepsilon > 0$ and $k_{max} > 0$

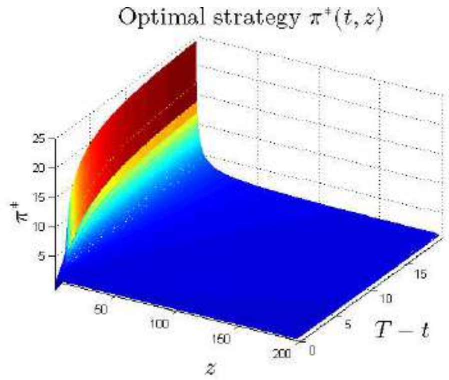
ii) Solve the linear PDE for $J^{(k)} = J^{\pi^k}$

$$J_{\tau}^{(k)} = a(\tau,z,p)J_{zz}^{(k)} + b(\tau,z,p)J_z^{(k)} - c(\tau,z,p)J^{(k)}, \quad \text{with}$$
$$J^k(0,z) = U(z) = \frac{z^{\gamma}}{\gamma}.$$

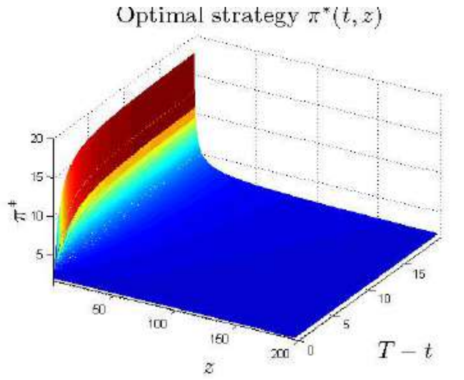
iii) Compute the improved control

$$p^{k+1}(\tau,z) = \arg\max_{p\in\mathbb{R}}\{a(\tau,z,p)J_{zz}^{(k)} + b(\tau,z,p)J_z^{(k)} - c(\tau,z,p)J^{(k)}\}.$$

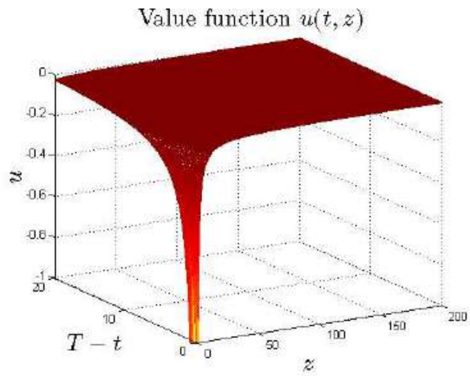
iv) If $\|J^{(k+1)} - J^{(k)}\| < \varepsilon$ or $\|p^{k+1} - p^k\| < \varepsilon$ or $k > k_{max}$, then stop else $k \leftarrow k+1$, go to $ii)$.



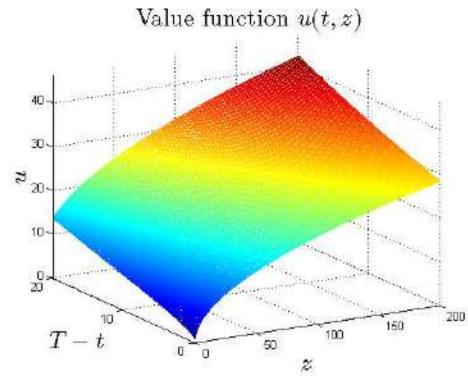
(a) Optimal strategy for $\gamma = -1$.



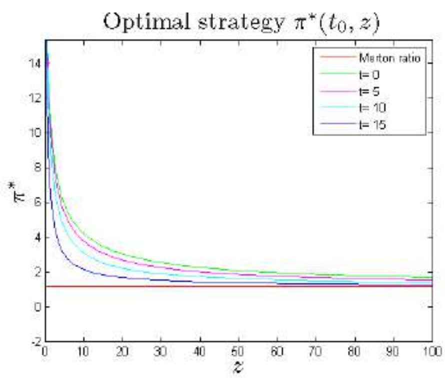
(b) Optimal strategy for $\gamma = 0.5$.



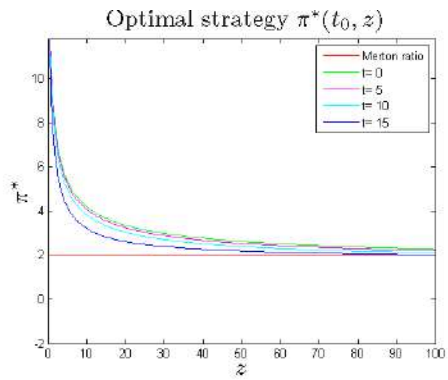
(c) Value function for $\gamma = -1$.



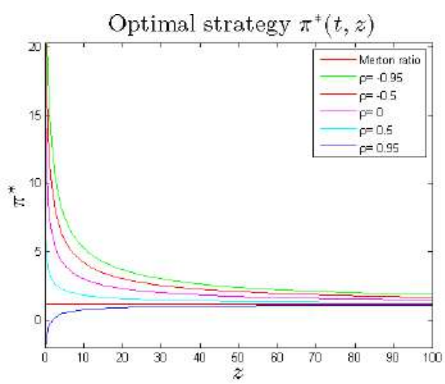
(d) Value function for $\gamma = 0.5$.



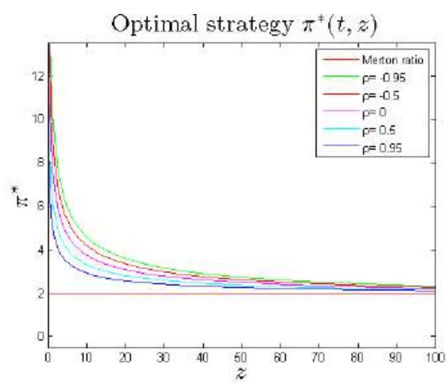
(e) Optimal strategy at different time points for $\gamma = -1$.



(f) Optimal strategy at different time points for $\gamma = 0.5$.

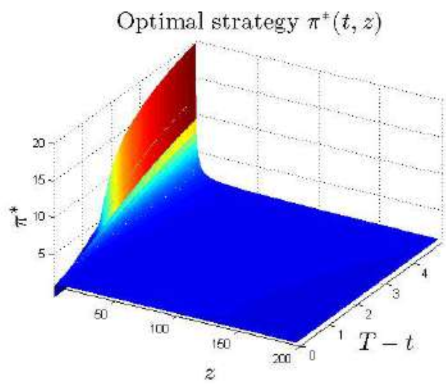


(g) Optimal strategy for different values of ρ with $\gamma = -1$.

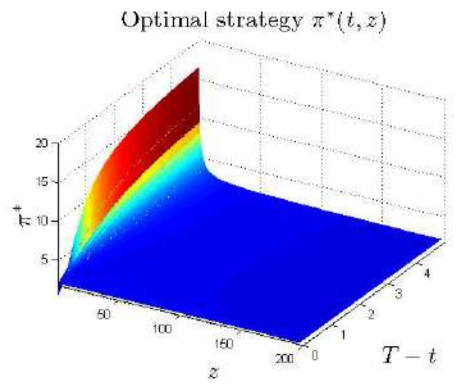


(h) Optimal strategy for different values of ρ with $\gamma = 0.5$.

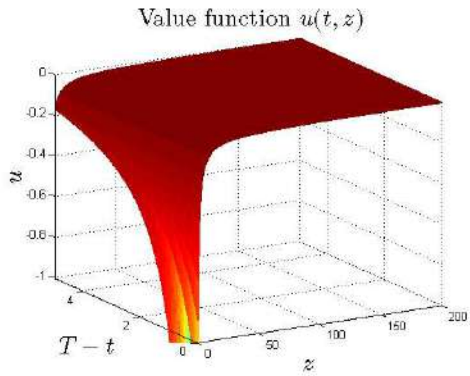
Figure 4.2: Numerical results for $T = 20$ years.



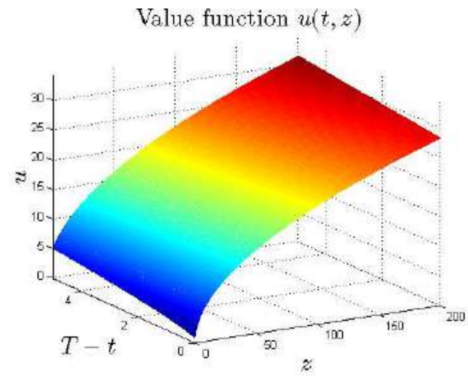
(a) Optimal strategy for $\gamma = -1$.



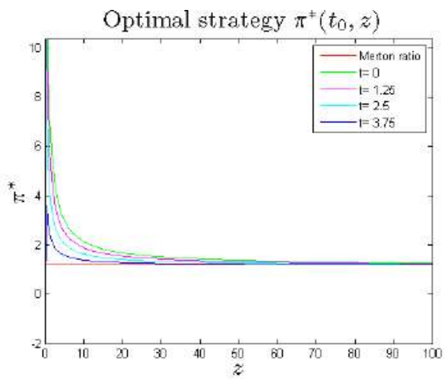
(b) Optimal strategy for $\gamma = 0.5$.



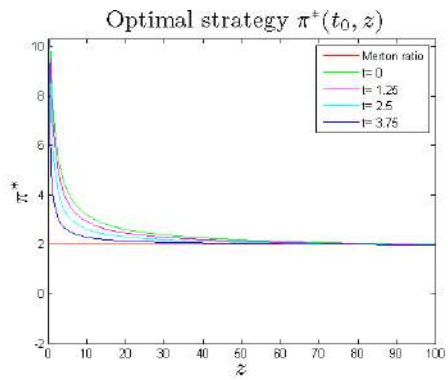
(c) Value function for $\gamma = -1$.



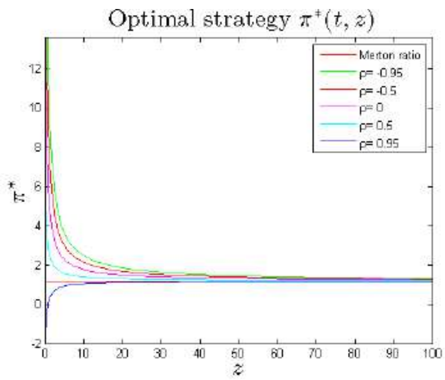
(d) Value function for $\gamma = 0.5$.



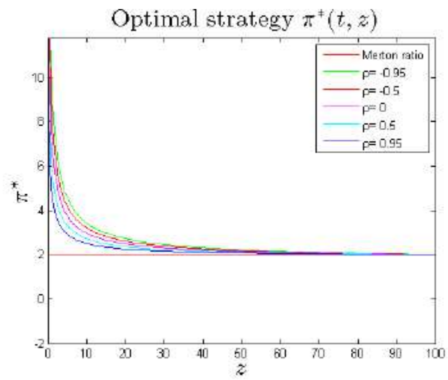
(e) Optimal strategy at different time points for $\gamma = -1$.



(f) Optimal strategy at different time points for $\gamma = 0.5$.

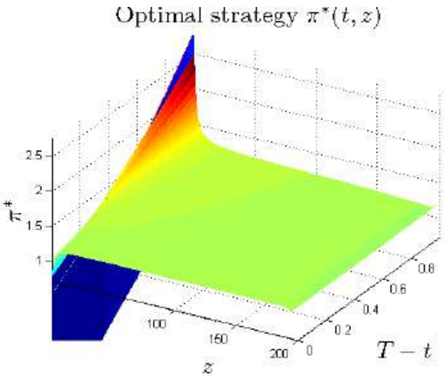


(g) Optimal strategy for different values of ρ with $\gamma = -1$.

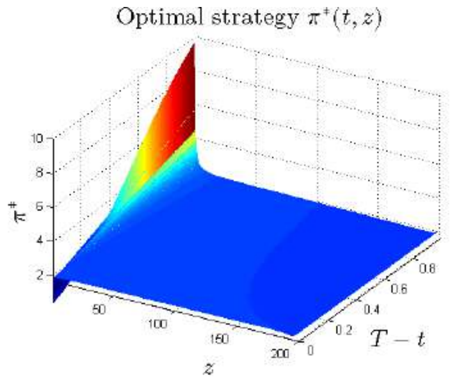


(h) Optimal strategy for different values of ρ with $\gamma = 0.5$.

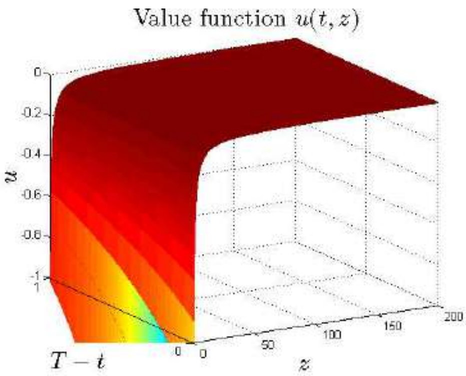
Figure 4.3: Numerical results for $T = 5$ years.



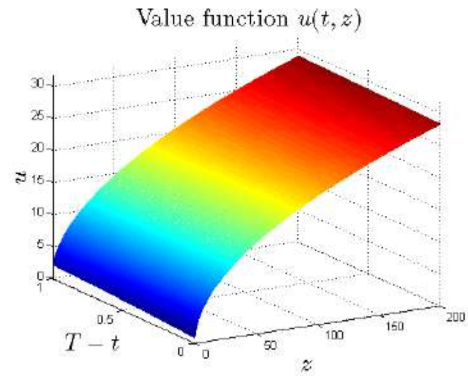
(a) Optimal strategy for $\gamma = -1$.



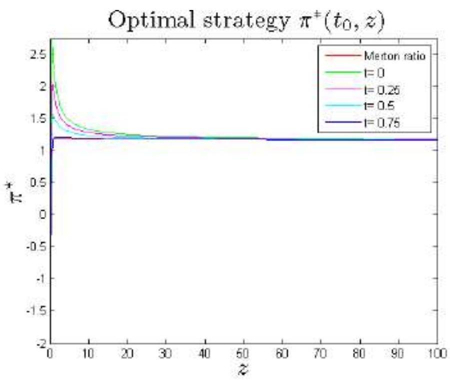
(b) Optimal strategy for $\gamma = 0.5$.



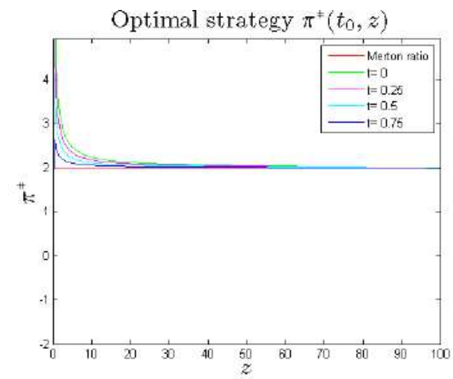
(c) Value function for $\gamma = -1$.



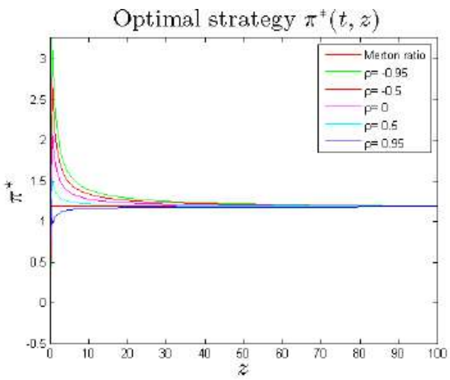
(d) Value function for $\gamma = 0.5$.



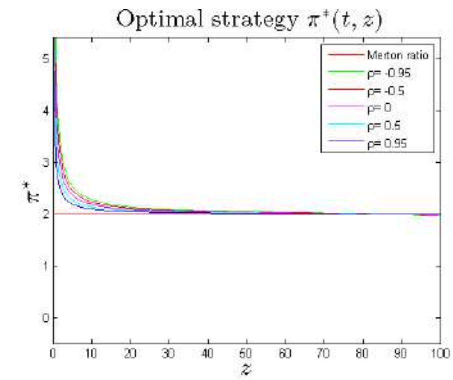
(e) Optimal strategy at different time points for $\gamma = -1$.



(f) Optimal strategy at different time points for $\gamma = 0.5$.

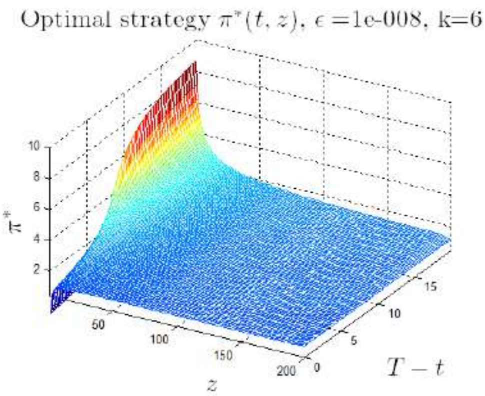


(g) Optimal strategy for different values of ρ with $\gamma = -1$.

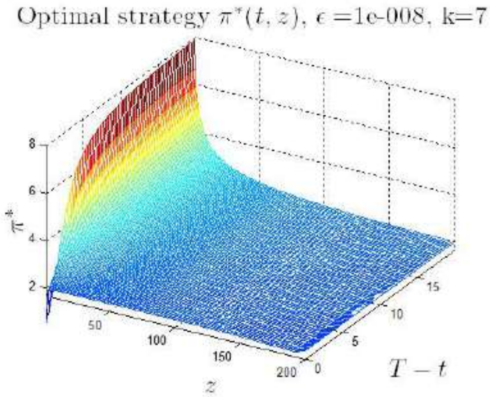


(h) Optimal strategy for different values of ρ with $\gamma = 0.5$.

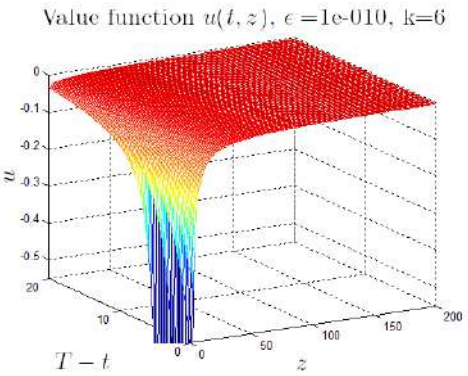
Figure 4.4: Numerical results for $T = 1$ year.



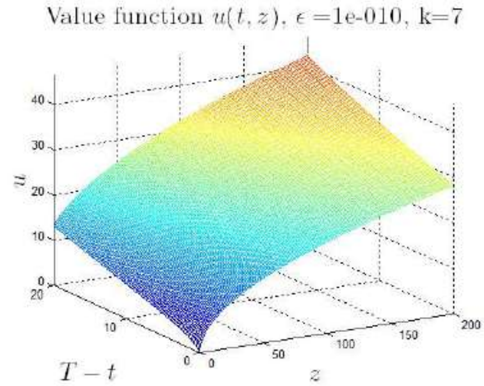
(a) Optimal strategy for $\gamma = -1$.



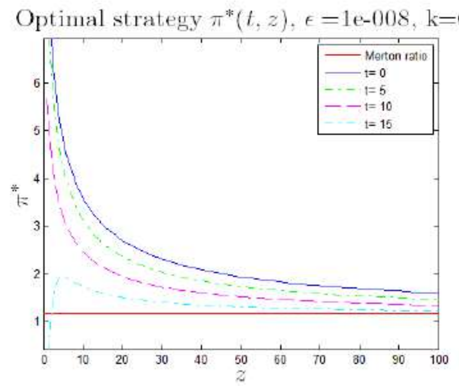
(b) Optimal strategy for $\gamma = 0.5$.



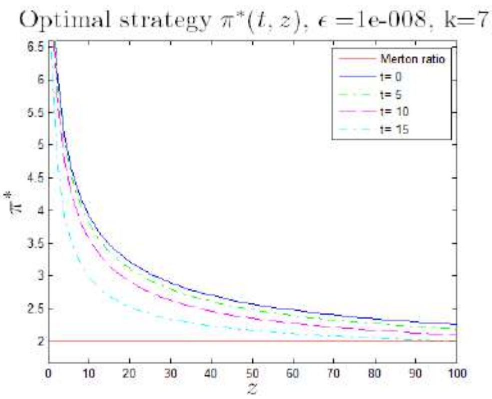
(c) Value function for $\gamma = -1$.



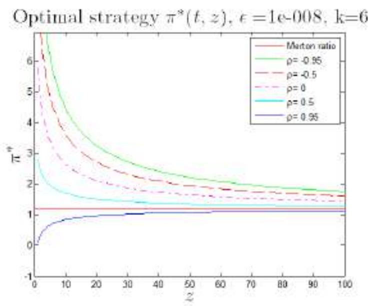
(d) Value function for $\gamma = 0.5$.



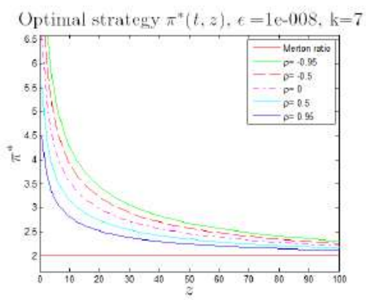
(e) Optimal strategy at different time points for $\gamma = -1$.



(f) Optimal strategy at different time points for $\gamma = 0.5$.



(g) Optimal strategy for different values of ρ with $\gamma = -1$.



(h) Optimal strategy for different values of ρ with $\gamma = 0.5$.

Figure 4.5: Numerical results for the policy improvement algorithm for $T = 20$ years, $N_\tau = 7300$, $k_{max} = 1000$, $N_z = 100$.

5. Conclusion

We have investigated a problem of optimal investment of an economic agent under stochastic endowments for a finite time period. The problem has been treated as a stochastic optimal control problem. We have investigated the associated HJB equation by means of viscosity solutions, giving a characterization of the value function as unique viscosity solution of the HJB equation. This has allowed us to use the finite difference method to compute the value function and see, numerically, what is the impact of the random income on the optimal value of the problem, and what is the optimal strategy. We have also been able to describe the asymptotic behaviour of the value function, and the optimal strategy when the initial wealth goes to zero or to infinity. We cannot reproduce the numerical results from [2] perfectly.

For further study it would be interesting to use alternative solution techniques which might confirm our results or those of [2]. For instance we have attempted with the policy improvement algorithm and the result is similar than what we got previously.

Appendix A. Proof of Theorem 3.2.1

Proof. To prove Theorem 3.2.1, we prove that v is both subsolution and supersolution. To show that v is sub (resp. super) solution, we adjust the method developed in [14, Proposition 4.3.2] for the case of an n-dimensional controlled state process and optimization of consumption problem in infinite-time horizon (resp. [14, Proposition 4.3.1] for the case of n-dimensional optimization of consumption in finite-time horizon), to the case of a 2-dimensional controlled state process (X_t^π, c_t) and optimal terminal wealth problem in finite-time horizon.

We know from the proposition (3.1.2) that v is continuous in the interior of the domain, hence locally bounded. We want to use this argument to show the following statements:

i) The value function v is a viscosity supersolution of (3.2.1) on $\mathcal{O}$.
Let $(\bar{t}, \bar{x}, \bar{y}) \in \mathcal{O}$ and $\varphi \in C^{1,2}(\mathcal{O})$ be a test function such that

$$0 = (v_* - \varphi)(\bar{t}, \bar{x}, \bar{y}) = \min_{(t,x,y)\in\mathcal{O}} (v_* - \varphi)(t, x, y). \tag{A.0.1}$$

By the definition of $v_*(\bar{t}, \bar{x}, \bar{y})$, there exists a sequence (t_n, x_n, y_n) in $\mathcal{O}$ such that

$$(t_n, x_n, y_n) \longrightarrow (\bar{t}, \bar{x}, \bar{y}) \text{ and } v(t_n, x_n, y_n) \longrightarrow v_*(\bar{t}, \bar{x}, \bar{y}), \text{ as } n \longrightarrow \infty.$$

By the continuity of φ and by (A.0.1), we also have

$$\gamma_n := v(t_n, x_n, y_n) - \varphi(t_n, x_n, y_n) \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Let $\pi \in \mathcal{A}(t_n, x_n, y_n) = \mathcal{A}_1$ be a control and denote by $(X_s^{\pi,t_n,x_n,y_n}, c_s^{t_n,y_n})$ the associate controlled process. Let $\tau_n = \inf\left\{ s \geq t_n : |(X_s^{\pi,t_n,x_n,y_n}, c_s^{t_n,y_n}) - (x_n, y_n)| \geq \delta \right\}$ be a stopping time in which $\delta > 0$ is a fixed constant. Let (h_n) be a strictly positive sequence such that

$$h_n \longrightarrow 0 \text{ and } \frac{\gamma_n}{h_n} \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{A.0.2}$$

We apply the first part of the dynamic programming principle (see [14, Section 3.3]) for $v(t_n, x_n, y_n)$ to $\beta_n = \tau_n \wedge (t_n + h_n)$ and obtain

$$v(t_n, x_n, y_n) \geq \mathbb{E}\left[v(\beta_n, X_{\beta_n}^{\pi,t_n,x_n,y_n}, c_{\beta_n}^{t_n,y_n}) \right]. \tag{A.0.3}$$

Equation (A.0.1) implies that $v \geq v_* \geq \varphi$, so that (A.0.3) becomes

$$\varphi(t_n, x_n, y_n) + \gamma_n \geq \mathbb{E}\left[\varphi(\beta_n, X_{\beta_n}^{\pi,t_n,x_n,y_n}, c_{\beta_n}^{t_n,y_n}) \right].$$

After applying the Dynkin's formula, we divide by h_n and move h_n within the expectation to obtain

$$\frac{\gamma_n}{h_n} + \mathbb{E}\left[\frac{1}{h_n} \int_{t_n}^{\beta_n} \left(-\frac{\partial \varphi}{\partial t} - \mathcal{L}^\pi \varphi \right) (s, X_s^{\pi,t_n,x_n,y_n}, c_s^{t_n,y_n}) ds \right] \geq 0. \tag{A.0.4}$$

By almost surely (a.s) continuity of the trajectory $(X_s^{\pi,t_n,x_n,y_n}, c_s^{t_n,y_n})$, it follows that for n sufficiently large ($n \geq N(w)$), $\beta_n = t_n + h_n \quad a.s.$ Thus, applying the mean value theorem together with equation (A.0.2) to equation (A.0.4), we get

$$\left(-\frac{\partial \varphi}{\partial t} - \mathcal{L}^\pi \varphi \right) (\bar{t}, \bar{x}, \bar{y}) \geq 0 \Longrightarrow F(\bar{t}, \bar{x}, \bar{y}, v_*(\bar{t}, \bar{x}, \bar{y}), \varphi_t(\bar{t}, \bar{x}, \bar{y}), D\varphi(\bar{t}, \bar{x}, \bar{y}), D^2\varphi(\bar{t}, \bar{x}, \bar{y})) \geq 0.$$

This shows that v is a viscosity supersolution of (3.2.1) on $\mathcal{O}$.

ii) The value function v is a viscosity subsolution of (3.2.1) on $\mathcal{O}$.
Let $(\bar{t}, \bar{x}, \bar{y}) \in \mathcal{O}$ and $\psi \in C^{1,2}(\mathcal{O})$ be a test function such that

$$0 = (v^* - \psi)(\bar{t}, \bar{x}, \bar{y}) = \max_{(t,x,y)\in\mathcal{O}} (v^* - \psi)(t, x, y). \quad \text{(A.0.5)}$$

By the definition of $v^*(\bar{t}, \bar{x}, \bar{y})$, there exists a sequence (t_n, x_n, y_n) in $\mathcal{O}$ such that

$$(t_n, x_n, y_n) \longrightarrow (\bar{t}, \bar{x}, \bar{y}) \text{ and } v(t_n, x_n, y_n) \longrightarrow v^*(\bar{t}, \bar{x}, \bar{y}), \quad as \quad n \longrightarrow \infty.$$

By the continuity of ψ and by (A.0.5), we also have

$$\theta_n := v(t_n, x_n, y_n) - \psi(t_n, x_n, y_n) \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

Let $\pi \in \mathcal{A}(t_n, x_n, y_n) = \mathcal{A}_1$ be a control and denote by $(X_s^{\pi,t_n,x_n,y_n}, c_s^{t_n,y_n})$ the associate controlled process. Let $\tau_n = \inf \left\{ s \geq t_n : |(X_s^{\pi,t_n,x_n,y_n}, c_s^{t_n,y_n}) - (x_n, y_n)| \geq \eta \right\}$ be a stopping time in which $\eta > 0$ is a fixed constant. Let (h_n) be a strictly positive sequence such that

$$h_n \longrightarrow 0 \text{ and } \frac{\theta_n}{h_n} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty. \quad \text{(A.0.6)}$$

We apply the second part of the dynamic programming principle (see [14, Section 3.3]) for $v(t_n, x_n, y_n)$ to $\lambda_n = \tau_n \wedge (t_n + h_n)$ and obtain for all $\varepsilon > 0$

$$v(t_n, x_n, y_n) - \varepsilon \frac{h_n^2}{2} \leq \mathbb{E}\left[v(\lambda_n, X_{\lambda_n}^{\pi,t_n,x_n,y_n}, c_{\lambda_n}^{t_n,y_n}) \right]. \quad \text{(A.0.7)}$$

Equation (A.0.5) implies that $v \leq v^* \leq \psi$, thus (A.0.7) becomes

$$\theta_n + \psi(t_n, x_n, y_n) - \varepsilon \frac{h_n^2}{2} \leq \mathbb{E}\left[\psi(\lambda_n, X_{\lambda_n}^{\pi,t_n,x_n,y_n}, c_{\lambda_n}^{t_n,y_n}) \right].$$

After applying the Dynkin's formula, we divide by h_n and move h_n within the expectation to obtain

$$\frac{\theta_n}{h_n} - \varepsilon \frac{h_n}{2} + \mathbb{E}\left[\frac{1}{h_n} \int_{t_n}^{\lambda_n} \left(-\frac{\partial \psi}{\partial t} - \mathcal{L}^\pi \psi \right) (s, X_s^{\pi,t_n,x_n,y_n}, c_s^{t_n,y_n}) ds \right] \leq 0. \quad \text{(A.0.8)}$$

By almost surely (a.s) continuity of the trajectory $(X_s^{\pi,t_n,x_n,y_n}, c_s^{t_n,y_n})$, it follows that for n sufficiently large $(n \geq N(w))$, $\lambda_n = t_n + h_n \quad a.s.$ Thus, applying the mean value theorem together with equation (A.0.6) to equation (A.0.8), we obtain

$$\left(-\frac{\partial \psi}{\partial t} - \mathcal{L}^\pi \psi \right) (\bar{t}, \bar{x}, \bar{y}) \geq 0 \Longrightarrow F(\bar{t}, \bar{x}, \bar{y}, v^*(\bar{t}, \bar{x}, \bar{y}), \psi_t(\bar{t}, \bar{x}, \bar{y}), D\psi(\bar{t}, \bar{x}, \bar{y}), D^2\psi(\bar{t}, \bar{x}, \bar{y})) \leq 0.$$

This shows that v is a viscosity subsolution of (3.2.1) on $\mathcal{O}$. Hence v is a viscosity solution of the HJB equation (3.2.1).

□

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