Nonlocal Characterization of Sobolev Spaces and Convergence of solutions to Elliptic Integrodiferential Equations

Guy Fabrice Foghem Gounoue

Technische Universtät Dresden

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Outline

1. BBM-Characterization of Sobolev spaces
2. (Non)local Neumann problems
3. Convergence from elliptic IDEs to elliptic PDEs

Part I:
Bourgain-Brezis-Mironescu characterization of Sobolev spaces

Motivation

In 2001, **Bourgain-Brezis-Mironescu (BBM)**, motivated by the study of**the asymptotic behavior** of the fractional Gagliardo-Nirenberg $W^{s,p}$-norm when $1 < p < \infty$ is fixed and $s \to 1^-$, $0 < s < 1$, showed that

$$\lim_{s\to 1^-}(1-s)\iint\limits_{\Omega\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\mathrm{d}y\mathrm{d}x = \frac{|\mathbb{S}^{d-1}|}{p}K_{d,p}\int_\Omega|\nabla u(x)|^p\mathrm{d}x, \quad u \in L^p(\Omega). \tag{1}$$

when $\Omega \subset \mathbb{R}^d$ is **an open bounded Lipschitz domain**. Here, ∇u is he distributional gradient of u and $K_{d,p}$ is the universal constant,

$$K_{d,p} = \fint_{\mathbb{S}^{d-1}}|w\cdot e|^p\mathrm{d}\sigma_{d-1}(w) = \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{d+p}{2}\right)\Gamma\left(\frac{1}{2}\right)}, \qquad \text{for some } e \in \mathbb{S}^{d-1}. \tag{2}$$

Intuitively, the relation (1) makes sense since the fractional Sobolev space

$$W^{s,p}(\Omega) = \left[L^p(\Omega), W^{1,p}(\Omega)\right]_{s,p}$$

is the interpolation space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$.

Motivation

Later, **Mazy'a & Shaposhnikova, 2002**, completed the asymptotic when $s \to 0^+$,

$$\lim_{s\to 0^+} s \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} \mathrm{d}y\mathrm{d}x = \frac{2|\mathbb{S}^{d-1}|}{p} \int_{\mathbb{R}^d} |u(x)|^p \mathrm{d}x, \qquad \text{for } u \in \bigcup_{0<s<1} W^{s,p}(\mathbb{R}^d).$$

More generally, **BBM** proved that the relation (1) generalizes as follows

$$\lim_{\varepsilon\to 0^+} \iint\limits_{\Omega\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_\varepsilon(x-y)\mathrm{d}y\mathrm{d}x = K_{d,p} \int_\Omega |\nabla u(x)|^p \mathrm{d}x, \quad u \in L^p(\Omega). \tag{3}$$

where $(\rho_\varepsilon)_\varepsilon$ is **an approximation of the unity**, i.e., satisfies

$$\rho_\varepsilon \geq 0 \;\text{ is radial,} \quad \int_{\mathbb{R}^d} \rho_\varepsilon(h)\mathrm{d}h = 1 \quad \text{and} \quad \lim_{\varepsilon\to 0} \int_{|h|>\delta} \rho_\varepsilon(h)\mathrm{d}h = 0\,. \tag{4}$$

Motivation

Our main motivation here is two-fold: To extend the BBM result (3)

- for unbounded domains with extension property;
- for a general sequence approximating the unity.

To be more precise, if $\Omega \subset \mathbb{R}^d$ is an extension domain(eventually unbounded) and $(\nu_\varepsilon)_\varepsilon$ is a family of p-**Lévy integrable radial** functions $\nu_\varepsilon : \mathbb{R}^d \to [0, \infty]$ such that

$$\int_{\mathbb{R}^d} (1 \wedge |h|^p)\nu_\varepsilon(h)\mathrm{d}h = 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_{|h|>\delta} (1 \wedge |h|^p)\nu_\varepsilon(h)\mathrm{d}h = 0\,, \tag{5}$$

with $\min(1, |h|^p) = 1 \wedge |h|^p$, then we also have

$$\lim_{\varepsilon \to 0^+} \iint_{\Omega\Omega} |u(x) - u(y)|^p \nu_\varepsilon(x - y)\mathrm{d}y\mathrm{d}x = K_{d,p} \int_\Omega |\nabla u(x)|^p \mathrm{d}x, \quad u \in L^p(\Omega). \tag{6}$$

Remark 1.

Consider the fractional kernel,

$$\nu_\varepsilon(h) = a_{\varepsilon,d,p}|h|^{-d-p+\varepsilon} \qquad \textit{with } a_{\varepsilon,d,p} = \frac{\varepsilon(p-\varepsilon)}{p|\mathbb{S}^{d-1}|}$$

then $(\nu_\varepsilon)_\varepsilon$ *satisfies* (5) *but there is no family* $(\rho_\varepsilon)_\varepsilon$ *satisfying* (4) *such that* $\nu_\varepsilon(h) = |h|^{-p}\rho_\varepsilon(h)$ *for* $h \neq 0$. *This shows that the class* $(\nu_\varepsilon)_\varepsilon$ *is strictly lager than the class* $(\rho_\varepsilon)_\varepsilon$.

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then $(\nu_\varepsilon)_\varepsilon$ satisfies (5) but there is no family $(\rho_\varepsilon)_\varepsilon$ satisfying (4) such that $\nu_\varepsilon(h) = |h|^{-p}\rho_\varepsilon(h)$ for $h \neq 0$. This shows that the class $(\nu_\varepsilon)_\varepsilon$ is strictly lager than the class $(\rho_\varepsilon)_\varepsilon$.

Lemma 1.

Assume $\Omega \subset \mathbb{R}^d$ is open. $1 \leq p < \infty$ then for all $u \in C_c^1(\mathbb{R}^d)$ then

$$\lim_{\varepsilon \to 0^+} \int_\Omega |u(x) - u(y)|^p \nu_\varepsilon(x-y)\mathrm{d}y = K_{d,p}|\nabla u(x)|^p, \quad x \in \mathbb{R}^d. \tag{7}$$

The Lebesgue convergence dominated theorem implies.

Theorem 2.

Assume $\Omega \subset \mathbb{R}^d$ is open and $1 \leq p < \infty$ then for all $u \in C_c^1(\mathbb{R}^d)$ then

$$\lim_{\varepsilon \to 0^+} \iint\limits_{\Omega\Omega} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\mathrm{d}y\mathrm{d}x = K_{d,p} \int_\Omega |\nabla u(x)|^p \mathrm{d}x, \quad u \in C_c^1(\mathbb{R}^d). \tag{8}$$

Proof. Assume $\Omega \neq \mathbb{R}^d$, fix for $0 < \delta < 1 \wedge \operatorname{dist}(x, \partial\Omega)$ so that $B_\delta(x) \subset \Omega$

$$
\begin{aligned}
&\lim_{\varepsilon \to 0^+} \int_\Omega |u(x) - u(y)|^p \nu_\varepsilon(x-y)\mathrm{d}y \\
&= \lim_{\varepsilon \to 0^+} \int_{B_\delta(x)} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\mathrm{d}y \\
&+ \lim_{\varepsilon \to 0^+} \int\limits_{\Omega \cap |x-y| \geq \delta} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\mathrm{d}y \quad \Big[\leq 2^p \|u\|_\infty \int\limits_{|h| \geq \delta} \nu_\varepsilon(h)\mathrm{d}h \to 0\Big] \\
&= \lim_{\varepsilon \to 0^+} \int_{B_\delta(0)} |\nabla u(x) \cdot h|^p \nu_\varepsilon(h)\mathrm{d}h, \, \big[u(x+h) - u(x) = \nabla u(x) \cdot h + 0(1)\big] \\
&= \lim_{\varepsilon \to 0^+} |\mathbb{S}^{d-1}| \int_0^\delta r^{d+d-1} \nu_\varepsilon(r)\mathrm{d}r \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} |\nabla u(x) \cdot w|^p \sigma_{d-1}(w)\mathrm{d}w \\
&= \lim_{\varepsilon \to 0^+} \underbrace{\int_{B_\delta(0)} 1 \wedge |h|^p |\nu_\varepsilon(h)\mathrm{d}h}_{= 1 - \int\limits_{|h| \geq \delta} 1 \wedge |h|^p \nu_\varepsilon(h)\mathrm{d}h \to 1,} \quad \fint_{\mathbb{S}^{d-1}} |\nabla u(x) \cdot w|^p \sigma_{d-1}(w)\mathrm{d}w = K_{d,p} |\nabla u(x)|^p.
\end{aligned}
$$

The case $\Omega = \mathbb{R}^d$ follows by the same token, taking $\delta = 1$.

Theorem 3 (BBM, 2001// preprint: GF, 2020, Characterization of Sobolev spaces).

Assume $\Omega \subset \mathbb{R}^d$ is open, $1 \leq p < \infty$ and $u \in L^p(\Omega)$. If

$$A_p := \liminf_{\varepsilon \to 0^+} \iint_{\Omega\Omega} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\mathrm{d}y\mathrm{d}x < \infty \tag{9}$$

then

- $u \in W^{1,p}(\Omega)$ *and* $K_{d,p}\|\nabla u\|^p_{L^p(\Omega)} \leq A_p$, *for* $p \neq 1$;
- $u \in BV(\Omega)$ *and* $K_{1,d}|u|_{BV(\Omega)} \leq A_1$, *for* $p = 1$.

Recall $W^{1,p}(\Omega) = \{u \in L^p(\Omega) : s.t.\ \partial_{x_i} u$ weak derivative in $L^p(\Omega)\}$ and $W^{1,1}(\Omega) \subset BV(\Omega)$ and $BV(\Omega)$ stands for the **space of bounded variations** on Ω,

$$BV(\Omega) = \{u \in L^1(\Omega) : |u|_{BV(\Omega)} < \infty\}$$

where $|u|_{BV(\Omega)}$ is the total variation of the Radon measure $|\nabla u|(\cdot)$

$$|\nabla u|(\Omega) = |u|_{BV(\Omega)} := \sup\Big\{ \int_\Omega u(x)\,\mathrm{div}\,\phi(x)\mathrm{d}x : \ \phi \in C_c^\infty(\Omega, \mathbb{R}^d),\ \|\phi\|_{L^\infty(\Omega)} \leq 1\Big\}.$$

Warning! **The converse of Theorem 3 is not true in general.**

Example 4 (Counterexample.).

Consider $\Omega = (-1,0) \cup (0,1)$ and put $u(x) = -\frac{1}{2}$ if $x \in (-1,0)$ and $u(x) = \frac{1}{2}$ if $x \in (0,1)$. Clearly, $u \in W^{1,p}(\Omega)$ for all $1 \leq p < \infty$ with $\nabla u = 0$. If $1 < p < \infty$ and $s \geq 1/p$ then $\|u\|_{W^{s,p}(\Omega)} = \infty$, i.e., for $\nu_\varepsilon(h) \sim \varepsilon |h|^{1+(1-\varepsilon)p}$, $(\varepsilon = 1 - s)$ we have

$$u \in W^{1,p}(\Omega) \qquad \text{whereas} \qquad A_p := \liminf_{\varepsilon \to 0^+} \iint_{\Omega\Omega} |u(x) - u(y)|^p \nu_\varepsilon(x-y) \mathrm{d}y \mathrm{d}x = \infty. \quad (10)$$

Remark 2.

*The **main reason** is that, $\Omega = (-1,0) \cup (0,1)$ **is not a $W^{1,p}$-extension domain**. Indeed, assume $\overline{u} \in W^{1,p}(\mathbb{R})$ is an extension of u defined. In particular, $\overline{u} \in W^{1,p}(-1,1)$ and $\overline{u} = u$ on Ω. The distributional derivative of $\overline{u}$ on $(-1,1)$ is $\nabla \overline{u} = \delta_0$, the Dirac mass at the origin. This contradicts the fact that $\overline{u} \in W^{1,p}(\mathbb{R})$.*

Definition 5.

An open set $\Omega \subset \mathbb{R}^d$ is called a $W^{1,p}$-extension (resp. BV-extension) domain if there exists a linear operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ (resp. $E : BV(\Omega) \to BV(\mathbb{R}^d)$) and a constant $C := C(\Omega, d)$ such that

$$Eu\,|_\Omega = u \quad \text{and} \quad \|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{1,p}(\Omega)} \quad \text{for all} \quad u \in W^{1,p}(\Omega)$$
$$(\text{resp.} \quad Eu\,|_\Omega = u \quad \text{and} \quad \|Eu\|_{BV(\mathbb{R}^d)} \leq C\|u\|_{BV(\Omega)} \quad \text{for all} \quad u \in BV(\Omega)).$$

Remark 3.

- *According to **Piotr Hajłasz et al 2008,** $W^{1,p}$-extension domain Ω is necessarily a d-set, i.e., there $c > 0$ such that $|\Omega \cap B_r(x)| \geq cr^d$ for all $x \in \Omega$ and $0 < r < 1$. The Lebesgue differentiation theorem implies $|\partial\Omega| = 0$ ($\partial\Omega$ has zero Lebesgue measure). Therefore,*

$$\int_{\partial\Omega} |\nabla Eu| \mathrm{d}x = 0. \tag{11}$$

- *As a complement, we will assume that a BV-extension domain Ω also satisfies the boundary condition*

$$|\nabla Eu|(\partial\Omega) = \int_{\partial\Omega} \mathrm{d}|\nabla Eu| = 0. \tag{12}$$

Here is the converse of Theorem 3.

Theorem 6 (BBM, 2001// preprint: GF, 2020 ArXiv:2008.07631).

Let $\Omega \subset \mathbb{R}^d$ be an extension domain. Let $u \in L^p(\Omega)$, $1 < p < \infty$ or $u \in W^{1,1}(\Omega)$ then

$$\lim_{\varepsilon \to 0^+} \iint\limits_{\Omega\Omega} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\mathrm{d}y\mathrm{d}x = K_{d,p} \int_\Omega |\nabla u(x)|^p \mathrm{d}x. \tag{13}$$

with the convention that $\|\nabla u\|_{L^p(\Omega)} = \infty$ if $u \notin W^{1,p}(\Omega)$.

*If $p = 1$, $u \in BV(\Omega)$ and $|\nabla Eu|(\partial\Omega) = 0$ then (**Juan Dávila 2002**)*

$$\lim_{\varepsilon \to 0^+} \iint\limits_{\Omega\Omega} |u(x) - u(y)| \nu_\varepsilon(x-y)\mathrm{d}y\mathrm{d}x = K_{d,1} \int_\Omega \mathrm{d}|\nabla u(x)| = K_{d,1}|u|_{BV(\Omega)}. \tag{14}$$

Sketch of the proof.

- Assume $u \in W^{1,p}(\mathbb{R}^d)$ then for all $h \in \mathbb{R}^d$, one easily check that

$$\int_{\mathbb{R}^d} |u(x+h) - u(x)|^p \mathrm{d}x \leq 2^p (1 \wedge |h|^p) \|u\|^p_{W^{1,p}(\mathbb{R}^d)}, \tag{15}$$

given that, $\int_{\mathbb{R}^d} (1 \wedge |h|^p) \nu_\varepsilon(h) \mathrm{d}h = 1$, this implies

$$\iint_{\mathbb{R}^d\,\mathbb{R}^d} |u(x+h) - u(x)|^p \nu_\varepsilon(h) \mathrm{d}x\mathrm{d}h \leq 2^p \|u\|^p_{W^{1,p}(\mathbb{R}^d)}. \tag{16}$$

- Assume $u \in W^{1,p}(\Omega)$ and let $\bar{u} \in W^{1,p}(\mathbb{R}^d)$ **be an extension of** u then (16) implies

$$\begin{aligned}\iint_{\Omega\Omega} |u(x) - u(y)|^p \nu_\varepsilon(x-y) \mathrm{d}x\mathrm{d}y &\leq \iint_{\mathbb{R}^d\,\mathbb{R}^d} |\bar{u}(x+h) - \bar{u}(x)|^p \nu_\varepsilon(h) \mathrm{d}x\mathrm{d}h \\ &\leq 2^p \|\bar{u}\|^p_{W^{1,p}(\mathbb{R}^d)} \leq 2^p C^p \|u\|^p_{W^{1,p}(\Omega)}.\end{aligned}$$

- Now if $u, v \in W^{1,p}(\Omega)$, we put

$$U_\varepsilon(x,y) = |u(x) - u(y)| \nu_\varepsilon^{1/p}(x-y) \text{ and } V_\varepsilon(x,y) = |u(x) - u(y)| \nu_\varepsilon^{1/p}(x-y).$$

The foregoing implies

$$\Big| \|U_\varepsilon\|_{L^p(\Omega\times\Omega)} - \|V_\varepsilon\|_{L^p(\Omega\times\Omega)} \Big| \leq \|U_\varepsilon - V_\varepsilon\|_{L^p(\Omega\times\Omega)} \leq 2C\|u-v\|_{W^{1,p}(\Omega)}. \qquad (17)$$

According to (17) it is sufficient to prove the result for u **in a dense subset of** $W^{1,p}(\Omega)$**.**

▶ Since Ω is an extension domain, $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{1,p}(\Omega)$. The case $1 < p < \infty$ or $u \in W^{1,1}(\Omega)$ follows from Theorem 2.

▶ If $p = 1$ and $u \in BV(\Omega)$, similar argument can be applied by using the following approximation result.

Theorem 7 (c.f. Evans & Gariepy, p.172 or Ambrosio, Fusco & Pallara, Theorem 3.9).

Let $\Omega \subset \mathbb{R}^d$ be open. For every $u \in BV(\Omega)$, there exist functions $(u_n)_n$ in $BV(\Omega) \cap C^\infty(\Omega) = W^{1,1}(\Omega) \cap C^\infty(\Omega)$ such that

$$\|u_n - u\|_{L^1(\Omega)} \xrightarrow{n\to\infty} 0 \quad \textit{and} \quad \|\nabla u_n\|_{L^1(\Omega)} \xrightarrow{n\to\infty} |u|_{BV(\Omega)}.$$

Warning! Theorem 7 **does not say** $BV(\Omega) \cap C^\infty(\Omega)$ is dense in $BV(\Omega)$.

Theorem 8 (BBM, '01 & Augusto Ponce, '04 & GF, '20, Asymptotic compactness).

Assume $\Omega \subset \mathbb{R}^d$ is open, bounded and Lipschitz. Let the family $(u_\varepsilon)_\varepsilon$ such that

$$\sup_{\varepsilon>0}\Big(\|u_\varepsilon\|_{L^p(\Omega)} + \iint_{\Omega\Omega} |u_\varepsilon(x) - u_\varepsilon(y)|^p \nu_\varepsilon(x-y)\mathrm{d}y\mathrm{d}x\Big) < \infty.$$

There is a subsequence $(\varepsilon_n)_n$ with $\varepsilon_n \to 0^+$ as $n \to \infty$ and a function $u \in L^p(\Omega)$ such that $\|u_{\varepsilon_n} - u\|_{L^p(\Omega)} \xrightarrow{n\to\infty} 0$. Moreover, $u \in W^{1,p}(\Omega)$ if $1 < p < \infty$ or $u \in BV(\Omega)$ if $p = 1$.

As a direct consequence of this we have.

Theorem 9 (Robust Poincaré inequality).

Assume $\Omega \subset \mathbb{R}^d$ is open, bounded and Lipschitz. There exist $C > 0$ and $\varepsilon_0 > 0$ such that

$$\|u - \fint_\Omega u\|^p_{L^p(\Omega} \leq C \iint_{\Omega\Omega} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\mathrm{d}x\mathrm{d}y, \quad \textit{for all } u \in L^p(\Omega) \textit{ and } \varepsilon \in (0, \varepsilon_0).$$

Indeed, letting $\varepsilon \to 0^+$ (using Theorem 6) implies the classical Poincaré inequality

$$\|u - \fint_\Omega u\|^p_{L^p(\Omega} \leq CK_{d,p} \int_\Omega |\nabla u(x)|^p \mathrm{d}x, \quad \text{for all } u \in L^p(\Omega). \tag{18}$$

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Part II:
Convergence of IDEs with Neumann condition

From now on, $p = 2$ and $\varepsilon = 2 - \alpha \in (0, 2)$.

- Let $(\nu_\alpha)_{\alpha\in(0,2)}$ be a family of functions such that for every α, and $\delta > 0$

$$\nu_\alpha \geq 0 \text{ is radial}, \quad \int_{\mathbb{R}^d} (1 \wedge |h|^2)\nu_\alpha(h)\mathrm{d}h = 1, \quad \lim_{\alpha\to 2} \int_{|h|>\delta} (1 \wedge |h|^2)\nu_\alpha(h)\mathrm{d}h = 0\,. \tag{19}$$

- For symmetric kernels $J^\alpha : \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag} \to [0, \infty], \alpha \in (0, 2)$, we set the conditions:

(E) **Elliptic condition:** There exists a constant $\Lambda \geq 1$ such that for every $\alpha \in (0, 2)$ and all $x, y \in \mathbb{R}^d$, with $x \neq y$

$$\Lambda^{-1}\nu_\alpha(x - y) \leq J^\alpha(x, y) \leq \Lambda\nu_\alpha(x - y). \tag{E}$$

(I) For each $\alpha \in (0, 2)$ the kernel J^α is **translation invariant**, i.e., for every $h \in \mathbb{R}^d$

$$J^\alpha(x + h, y + h) = J^\alpha(x, y). \tag{I}$$

- Introduce the Lévy type Integro-differential operator L_α and $\mathcal{N}_\alpha$ the nonlocal normal derivative across Ω defined by

$$L_\alpha u(x) = p.v.2 \int_{\mathbb{R}^d} (u(x) - u(y))J^\alpha(x, y)\mathrm{d}y, \quad (x \in \mathbb{R}^d) \tag{20}$$

$$\mathcal{N}_\alpha u(x) = 2 \int_\Omega (u(x) - u(y))J^\alpha(x, y)\mathrm{d}y, \qquad (x \in \Omega^c). \tag{21}$$

- We define **the symmetric matrix** $A(x) = (a_{ij}(x))_{1\leq i,j\leq d}$ by

$$a_{ij}(x) = \lim_{\alpha\to 2}\int_{B_\delta(0)} h_i h_j J^\alpha(x, x+h)dh \quad \text{for } x\in\mathbb{R}^d \text{ and } \delta>0. \tag{22}$$

- The condition (E), implies that the matrix $A(x) = (a_{ij}(x))_{1\leq i,j\leq d}$, is **elliptic** and,

$$d^{-1}\Lambda^{-1}|\xi|^2 \leq A(x)\xi\cdot\xi \leq d^{-1}\Lambda|\xi|^2 \qquad \text{for every } x,\xi\in\mathbb{R}^d. \tag{23}$$

- Under the condition (I), the matrix A is constant. In particular

• for $J^\alpha(x,y) = \nu_\alpha(x-y)$ we have $A(x) = \frac{1}{d}(\delta_{ij})_{ij} = \frac{1}{d}I_d$.

• for $J^\alpha(x,y) = C_{d,\alpha}|x-y|^{-d-\alpha}$ we have $A(x) = (\delta_{ij})_{ij} = I_d$ and $L_\alpha = (-\Delta)^{\alpha/2}$ is **the fractional Laplacian**

$$(-\Delta)^{\alpha/2}u(x) = C_{d,\alpha}p.v.\int_{\mathbb{R}^d}\frac{(u(x)-u(y))}{|x-y|^{d+\alpha}}\mathrm{d}y. \tag{24}$$

here $C_{d,\alpha}\asymp\alpha(2-\alpha)$ is a normalization constant.

(Non)local Neumann problem

▶ **Local Green-Gauss formula:** Assume Ω is Lipschitz bounded and A is any elliptic matrix.

$$-\int_\Omega \operatorname{div}(A(\cdot)\nabla u)v \,\mathrm{d}x = \mathcal{E}^A(u,v) - \int_{\partial\Omega} \frac{\partial u}{\partial n_A} v \,\mathrm{d}\sigma(x) \qquad u, v \in C^2(\mathbb{R}^d). \tag{G_0}$$

here $\frac{\partial u(x)}{\partial n_A} = A(x)\nabla u(x) \cdot n(x)$ **is the outer normal derivative of** u **on** $\partial\Omega$ **w.r.t.** A and

$$\mathcal{E}^A(u,v) = \int_\Omega A(x)\nabla u(x) \cdot \nabla v(x) \mathrm{d}x. \tag{25}$$

Definition 10 (weak solution to local Neumann problem).

Let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be measurable. We say that $u : \Omega \to \mathbb{R}$ is a **weak solution** of the local Neumann problem

$$-\operatorname{div}(A(\cdot)\nabla u) = f \quad \text{in} \quad \Omega \qquad \text{and} \qquad \frac{\partial u}{\partial n_A} = g \quad \text{on} \quad \partial\Omega, \tag{N_0}$$

if $u \in H^1(\Omega) = W^{1,2}(\Omega)$ and

$$\mathcal{E}^A(u,v) = \int_\Omega f(x)v(x)\mathrm{d}x + \int_{\partial\Omega} g(x)v(x)\mathrm{d}\sigma(x), \quad \text{for all} \quad v \in H^1(\Omega). \tag{V_0}$$

Taking $v = 1$, (V_0) gives the **compatibility condition**

$$\int_\Omega f(x)\mathrm{d}x + \int_{\partial\Omega} g(y)\mathrm{d}y = 0. \tag{C_0}$$

Nonlocal Neumann problem

- **Nonlocal Green-Gauss formula:**

$$\int_{\Omega} L_\alpha u(x)v(x)\mathrm{d}x = \mathcal{E}^\alpha(u,v) - \int_{\Omega^c} \mathcal{N}_\alpha u(y)v(y)\mathrm{d}y \qquad u,v \in C_c^\infty(\mathbb{R}^d). \tag{G_α}$$

Recall that

$$L_\alpha u(x) = \mathrm{p.\,v.}\, 2\int_{\mathbb{R}^d} (u(x)-u(y))J^\alpha(x,y)\mathrm{d}y, \qquad (x \in \mathbb{R}^d) \tag{26}$$

$$\mathcal{N}_\alpha u(x) = 2\int_{\Omega} (u(x)-u(y))J^\alpha(x,y)\mathrm{d}y, \qquad (x \in \Omega^c) \tag{27}$$

$$\mathcal{E}^\alpha(u,v) = \iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x)-u(y))(v(x)-v(y))J^\alpha(x,y)\mathrm{d}x\mathrm{d}y. \tag{28}$$

Note that $(\Omega^c \times \Omega^c)^c = (\mathbb{R}^d \times \mathbb{R}^d) \setminus (\Omega^c \times \Omega^c)$.

- Define

$$V_{\nu_\alpha}(\Omega|\,\mathbb{R}^d) = \Big\{u : \mathbb{R}^d \to \mathbb{R} \text{ meas. : } u\,|_\Omega \in L^2(\Omega) \text{ and } \mathcal{E}^\alpha(u,u) < \infty\Big\}. \tag{29}$$

If ν_α has full support then $V_{\nu_\alpha}(\Omega|\,\mathbb{R}^d)$ is a Hilbert space under the norm

$$\|u\|^2_{V_{\nu_\alpha}(\Omega|\,\mathbb{R}^d)} = \|u\|^2_{L^2(\Omega)} + \iint\limits_{(\Omega^c\times\Omega^c)^c} (u(x)-u(y))^2\nu_\alpha(x-y)\mathrm{d}x\mathrm{d}y. \tag{30}$$

Definition 11 (Dipierro et al 2017: weak solution to nonlocal Neumann problem).

Let $f_\alpha : \Omega \to \mathbb{R}$ and $g_\alpha : \Omega^c \to \mathbb{R}$ be measurable. We say that $u_\alpha : \mathbb{R}^d \to \mathbb{R}$ is a **weak solution** of the nonlocal Neumann problem

$$L_\alpha u_\alpha = f_\alpha \quad \text{in} \quad \Omega \quad \text{and} \quad \mathcal{N}_\alpha u_\alpha = g_\alpha \quad \text{on} \quad \Omega^c \tag{N_α}$$

if $u \in V_{\nu_\alpha}(\Omega|\,\mathbb{R}^d)$ and

$$\mathcal{E}^\alpha(u_\alpha, v) = \int_\Omega f_\alpha(x)v(x)\mathrm{d}x + \int_{\Omega^c} g_\alpha(y)v(y)\mathrm{d}y, \quad \text{for all} \quad v \in V_{\nu_\alpha}(\Omega|\,\mathbb{R}^d)\,. \tag{V_α}$$

Taking $v = 1$, (V_α) becomes the so called **compatibility condition**

$$\int_\Omega f_\alpha(x)\mathrm{d}x + \int_{\Omega^c} g_\alpha(y)\mathrm{d}y = 0. \tag{C_α}$$

Well-posedness of (non)local Neumann problem

Theorem 12 (Local Neumann problem).

Assume Ω is bounded Lipschitz, $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$.

- *There exists a unique $u \in H^1(\Omega)^\perp = \left\{v \in H^1(\Omega) : \fint_\Omega v = 0\right\}$ satisfying the* ***modified Neumann problem***

$$\mathcal{E}^A(u,v) = \int_\Omega f(x)v(x)\mathrm{d}x + \int_{\partial\Omega} g(y)v(y)\mathrm{d}\sigma(y) \quad \text{for all} \quad v \in H^1(\Omega)^\perp. \qquad (V_0')$$

- *If f and g are* ***compatible****, solutions to local Neumann problem (V_0)* ***are of the form***

$$w = u + c \in H^1(\Omega), \qquad c \in \mathbb{R}.$$

- *Moreover, all $w's$ satisfy the weak regularity estimate,*

$$\|w - \fint_\Omega w\|_{V_\nu(\Omega|\mathbb{R}^d)} \leq C\Big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\Big). \qquad (31)$$

with the constant $C := C(d,\Omega) > 0$.

Theorem 13 (Nonlocal Neumann problem).

*Assume ν_α is **almost decreasing**, i.e., $\nu_\alpha(|x|) \leq \kappa \nu_\alpha(|y|)$ if $|x| \geq |y|$ for some $\kappa > 0$, $f_\alpha \in L^2(\Omega)$ and $g_\alpha \in L^2(\Omega^c, \nu^{-1}_{\alpha,\Omega})$, here we let*

$$\nu_{\alpha,\Omega}(x) = \operatorname{essinf}_{y\in\Omega} \nu_\alpha(x-y).$$

- *There exists a unique $u_\alpha \in V_{\nu_\alpha}(\Omega|\mathbb{R}^d)^\perp = \left\{v \in V_{\nu_\alpha}(\Omega|\mathbb{R}^d) : \fint_\Omega v = 0\right\}$ satisfying the **modified Neumann problem***

$$\mathcal{E}^\alpha(u_\alpha, v) = \int_\Omega f_\alpha(x)v(x)\mathrm{d}x + \int_{\Omega^c} g_\alpha(y)v(y)\mathrm{d}y \quad \text{for all} \;\; v \in V_{\nu_\alpha}(\Omega|\mathbb{R}^d)^\perp. \quad (V'_\alpha)$$

- *If f_α and g_α are **compatible**, solutions to nonlocal Neumann problem (V_α) **exist** and are of **the form***

$$w_\alpha = u_\alpha + c \in V_{\nu_\alpha}(\Omega|\mathbb{R}^d) \qquad c \in \mathbb{R}.$$

- *Moreover, all $w'_\alpha s$ satisfy the weak regularity estimate*

$$\|w_\alpha - \fint_\Omega w_\alpha\|_{V_{\nu_\alpha}(\Omega|\mathbb{R}^d)} \leq C\Big(\|f_\alpha\|_{L^2(\Omega)} + \|g_\alpha\|_{L^2(\Omega^c,\nu^{-1}_{\alpha,\Omega})}\Big). \qquad (32)$$

with the constant $C := C(d,\Omega,\nu_\alpha) > 0$.

From nonlocal to local

Analogous analysis to Theorem 6 lead to the following.

Theorem 14 (Bourgain-Brezis-Mironescu type result).

Let $D \subset \mathbb{R}^d$ be an extension domain. For all $u \in L^2(D)$ we have

$$\lim_{\alpha\to 2} \iint_{DD} (u(x)-u(y))^2 J^\alpha(x,y)\mathrm{d}x\mathrm{d}y = \int_D A(x)\nabla u(x)\cdot\nabla v(x)\mathrm{d}x,$$

$$\lim_{\alpha\to 2} \iint_{DD} (u(x)-u(y))^2 \nu_\alpha(x-y)\mathrm{d}x\mathrm{d}y = K_{d,2}\int_D |\nabla u(x)|^2\mathrm{d}x \qquad K_{d,2}=\frac{1}{d}.$$

Direct consequence: If $\Omega \subset \mathbb{R}^d$ is bounded Lipschitz: so that $\mathbb{R}^d$, Ω and Ω^c are extension domains,

$$\begin{aligned}
&\lim_{\alpha\to 2} \iint_{(\Omega^c\times\Omega^c)^c} (u(x)-u(y))^2 J^\alpha(x,y)\mathrm{d}x\mathrm{d}y \\
&= \lim_{\alpha\to 2}\Big[\iint_{\mathbb{R}^d\,\mathbb{R}^d} - \iint_{\Omega^c\Omega^c}\Big](u(x)-u(y))^2 J^\alpha(x,y)\mathrm{d}x\mathrm{d}y \\
&= \Big[\int_{\mathbb{R}^d} - \int_{\Omega^c}\Big] A(x)\nabla u(x)\cdot\nabla v(x)\mathrm{d}x = \int_\Omega A(x)\nabla u(x)\cdot\nabla v(x)\mathrm{d}x.
\end{aligned}$$

Some spin-offs:

- Therefore, for all $u \in H^1(\mathbb{R}^d)$,

$$\lim_{\alpha \to 2} \mathcal{E}^\alpha(u,u) = \mathcal{E}^A(u,u). \tag{33}$$

- If $(J^\alpha)_\alpha$ satisfies thee translation invariant condition (I), one can show

$$\lim_{\alpha \to 2} L_\alpha \varphi(x) = -\operatorname{div}(A(x)\nabla\varphi)(x) \qquad \varphi \in C_b^2(\mathbb{R}^d). \tag{34}$$

- The above implies that for $v \in H^1(\mathbb{R}^d)$, $g_\alpha = \mathcal{N}_\alpha \varphi$ and $g = \frac{\partial \varphi}{\partial n_A}$,

$$\int_{\partial\Omega} g(x)v(x)\mathrm{d}\sigma(x) = \lim_{\alpha \to 2} \int_{\Omega^c} g_\alpha(y)v(y)\mathrm{d}y. \tag{35}$$

Indeed, combining **the local and the nonlocal Green-Gauss** formula gives

$$\begin{aligned}
\int_{\partial\Omega} \frac{\partial\varphi(x)}{\partial n_A} v(x)\mathrm{d}\sigma(x) &= \mathcal{E}^A(\varphi, v) + \int_\Omega \operatorname{div}(A(x)\nabla\varphi(x))v(x)\mathrm{d}x \\
&= \lim_{\alpha \to 2} \mathcal{E}^\alpha(\varphi, u) - \int_\Omega L_\alpha\varphi(x)v(x)\mathrm{d}x \\
&= \lim_{\alpha \to 2} \int_{\Omega^c} \mathcal{N}_\alpha\varphi(y)v(y)\mathrm{d}y \qquad g_\alpha = \mathcal{N}_\alpha\varphi.
\end{aligned}$$

- Taking $J^\alpha(x,y) = C_{d,\alpha}|x-y|^{-d-\alpha}$ gives $L_\alpha = (-\Delta)^{\alpha/2}$, $-\operatorname{div}(A\nabla) = -\Delta$ and

$$\frac{\partial\varphi(x)}{\partial n_A} = \frac{\partial\varphi(x)}{\partial n} = \nabla\varphi(x)\cdot n(x).$$

▶ $(G_\alpha) \to (G_0)$ as $\alpha \to 2$, i.e., letting $\alpha \to 2$ in the nonlocal Green-Gauss formula (G_α)

$$\int_\Omega L_\alpha u(x)v(x)\mathrm{d}x = \mathcal{E}^\alpha(u,v) - \int_{\Omega^c} \mathcal{N}_\alpha u(y)v(y)\mathrm{d}y \tag{36}$$

one recovers the local local Green-Gauss formula (G_0)

$$-\int_\Omega \operatorname{div}(A(x)\nabla u(x))v(x)\mathrm{d}x = \mathcal{E}^A(u,v) - \int_{\partial\Omega} \frac{\partial u(x)}{\partial n_A} v(x)\mathrm{d}\sigma(x) \tag{37}$$

Furthermore we have the following more general convergence of the energies forms.

Theorem 15 (GF/Kassmann/Voigt '19: Mosco convergence).

Define $H_{\nu_\alpha}(\Omega) = \{u \in L^2(\Omega) : \mathcal{E}^\alpha_\Omega(u,u) < \infty\}$ *where*

$$\mathcal{E}^\alpha_\Omega(u,u) = \iint\limits_{\Omega\Omega} (u(x)-u(y))^2 J^\alpha(x,y)\mathrm{d}x\mathrm{d}y \tag{38}$$

Then, as $\alpha \to 2$, *both nonlocal forms*

$$(\mathcal{E}^\alpha(\cdot,\cdot), V_{\nu_\alpha}(\Omega|\mathbb{R}^d))_\alpha, (\mathcal{E}^\alpha_\Omega(\cdot,\cdot), H_{\nu_\alpha}(\Omega))_\alpha \xrightarrow{\text{Mosco converge}} (\mathcal{E}^A(\cdot,\cdot), H^1(\Omega)).$$

▶ *Note that **Mosco convergence** implies the **Gamma convergence.***

Theorem 16 (Convergence of Neumann problem).

Let Ω be bounded Lipschitz. Assume $(f_\alpha)_\alpha \rightharpoonup f$ (weakly) in $L^2(\Omega)$ as $\alpha \to 2$. Define

$$g_\alpha = \mathcal{N}_\alpha \varphi \quad \text{and} \quad g = \frac{\partial \varphi}{\partial n_A}, \qquad \text{with } \varphi \in C_b^2(\mathbb{R}^d).$$

*Assume the elliptic condition (E), and $u_\alpha \in V_{\nu_\alpha}(\Omega|\mathbb{R}^d)^\perp$ satisfies (**weak solution**) the nonlocal Neumann problem*

$$L_\alpha u_\alpha = f_\alpha \quad \text{on} \quad \Omega \quad \text{and} \quad \mathcal{N}_\alpha u_\alpha = g_\alpha \quad \text{on} \quad \Omega^c. \tag{39}$$

Let $u \in H^1(\Omega)^\perp$ be the unique weak solution in $H^1(\Omega)^\perp$ of the Neumann problem

$$-\operatorname{div}(A(\cdot)\nabla u) = f \quad \text{on} \quad \Omega \quad \text{and} \quad \frac{\partial u}{\partial n_A} = g \quad \text{on} \quad \partial\Omega. \tag{40}$$

*Assume that the condition (I) holds or that $g_\alpha = g = 0$ then $(u_\alpha)_\alpha$ converges to u in $L^2(\Omega)$, i.e., $\|u_\alpha - u\|_{L^2(\Omega)} \xrightarrow{\alpha\to 2} 0$. Moreover, we have **the weak convergence,***

$$\mathcal{E}^\alpha(u_\alpha, v) \xrightarrow{\alpha\to 2} \mathcal{E}^A(u, v) \qquad \text{for all } v \in H^1(\mathbb{R}^d). \tag{41}$$

Sketch of the proof.

▶ The **Robust Poincaré inequality** (Theorem 9) implies that, for $0 < \alpha_* < 2$ and $C > 0$,

$$\sup_{\alpha\in(\alpha_*,2)} \Big[\|u_\alpha\|^2_{L^2(\Omega)} + \mathcal{E}^\alpha(u_\alpha, u_\alpha)\Big] \leq C \tag{42}$$

▶ By the compactness (Theorem 8) there exists $\alpha_j \to 2$ and $u \in H^1(\Omega)^\perp$ such that

$$\|u_{\alpha_j} - u\|_{L^2(\Omega)} \xrightarrow{j\to\infty} 0$$

▶ In fact, for fixed $v \in H^1(\mathbb{R}^d)$ one can further establish the weak convergence

$$\mathcal{E}^{\alpha_j}(u_{\alpha_j}, v) \xrightarrow{\alpha_j\to 2} \mathcal{E}^A(u, v). \tag{43}$$

▶ We have previously shown that

$$\int_{\partial\Omega} g(x)v(x)\mathrm{d}\sigma(x) = \lim_{\alpha_j\to 2} \int_{\Omega^c} g_{\alpha_j}(y)v(y)\mathrm{d}y. \tag{44}$$

▶ By assumption $f_\alpha \rightharpoonup f$ in $L^2(\Omega)$.

Hence, for $u \in H^1(\mathbb{R}^d)$ we have

$$\begin{aligned}\int_\Omega f(x)v(x)\mathrm{d}\sigma(x) + \int_{\partial\Omega} g(x)v(x)\mathrm{d}\sigma(x) &= \lim_{\alpha_j \to 2} \int_\Omega f_{\alpha_j}(x)v(x)\mathrm{d} + \int_{\Omega^c} g_{\alpha_j}(y)v(y)\mathrm{d}y \\ &= \lim_{\alpha_j \to 2} \mathcal{E}^{\alpha_j}(u_{\alpha_j}, v) = \mathcal{E}^A(u, v).\end{aligned}$$

It turns out that

$$\mathcal{E}^A(u, v) = \int_\Omega f(x)v(x)\mathrm{d}\sigma(x) + \int_{\partial\Omega} g(x)v(x)\mathrm{d}\sigma(x).$$

that is, since A is elliptic and Ω is Lipschitz, $u \in H^1(\Omega)^\perp$ is the unique weak solution of the local Neumann problem. Thus the convergence of the entire sequence holds.

Theorem 17 (Convergence of Neumann eigenpairs).

Assume that $(\mu_\alpha, \phi_\alpha) \in \mathbb{R} \times L^2(\Omega)$ *is a* ***normalized Neumann eigenpairs*** *of the operator* L_α*, i.e.* $\|\phi_\alpha\|_{L^2(\Omega)} = 1$ *and we have*

$$L_\alpha \phi_\alpha, = \mu_\alpha \phi_\alpha \text{ on } \Omega \quad \text{and} \quad \mathcal{N}_\alpha \phi_\alpha = 0 \text{ on } \Omega^c. \tag{45}$$

Then there is $(\mu, \phi) \in \mathbb{R} \times L^2(\Omega)$ *such that, up to a subsequence,*

$$\mu_\alpha \xrightarrow{\alpha \to 2} \mu \quad \text{and} \quad \|\phi_\alpha - \phi\|_{L^2(\Omega)} \xrightarrow{\alpha \to 2} 0. \tag{46}$$

Moreover, (μ, ϕ) *is a* ***normalized Neumann eigenpairs*** *of the operator* $-\operatorname{div}(A\nabla)$*, i.e.*

$$-\operatorname{div}(A\nabla)\phi, = \mu\phi \text{ on } \Omega \quad \text{and} \quad \frac{\partial \phi}{\partial n_A} = 0 \text{ on } \partial\Omega. \tag{47}$$

Remark:

1. Theorem 16 and Theorem 17 remain true when **the Neumann condition** is replaced with **the Dirichlet condition.**
2. For $L_\alpha = (-\Delta)^{\alpha/2}$ or $J^\alpha(x,y) = d^{-1}\nu_\alpha(x-y)$, we have $-\operatorname{div}(A\nabla) = -\Delta$

$$L_\alpha u_\alpha = f_\alpha \xrightarrow{\alpha \to 2} -\Delta u = f, \qquad \textbf{(convergence of problems)}$$

$$L_\alpha \phi_\alpha = \lambda_\alpha \phi_\alpha \xrightarrow{\alpha \to 2} -\Delta\phi = \lambda\phi, \qquad \textbf{(convergence of eigen-problems)}.$$

Thank You For Your Attention.

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