Applicable Analysis
An International Journal

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gapa20

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To cite this article: L. L. Djomegne Njoukoue (2021): Hierarchic control for a nonlinear parabolic equation in an unbounded domain, Applicable Analysis, DOI: 10.1080/00036811.2021.1991327

To link to this article: https://doi.org/10.1080/00036811.2021.1991327

Published online: 15 Oct 2021.

APPLICABLE ANALYSIS
https://doi.org/10.1080/00036811.2021.1991327

Hierarchic control for a nonlinear parabolic equation in an unbounded domain

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ABSTRACT
In this paper, we study the hierarchical control using the Stackelberg–Nash strategy for a nonlinear parabolic equation in an unbounded domain. We assume that we can act on the system by three controls hierarchically. Two controls called followers that provide a Nash equilibrium for two cost functional. The third control named leader is supposed to bring the state of the system to rest at the final time. The results are achieved by means of observability inequality of Carleman type that we established for the adjoint systems and a fixed point theorem under the assumption that the uncontrolled domain is bounded.

ARTICLE HISTORY
Received 2 August 2021
Accepted 29 September 2021

COMMUNICATED BY
G. M. N'Guerekata

KEYWORDS
Nonlinear parabolic equation; Carleman inequality; Nash equilibrium; null controllability; Stackelberg–Nash strategy

MATHEMATICS SUBJECT CLASSIFICATIONS
49J20; 92D25; 93B05; 93C41

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ with $n \geq 1$ be an unbounded connected open set with boundary Γ at least of class $\mathcal{C}^2$ uniformly. Let ω, $\mathcal{O}_1$ and $\mathcal{O}_2$ be three nonempty subsets of Ω such that $\mathcal{O}_i \cap \omega = \emptyset$. For the time $T > 0$, we set $Q = (0,T) \times \Omega$, $\omega^T = (0,T) \times \omega$, $\mathcal{O}_1^T = (0,T) \times \mathcal{O}_1$, $\mathcal{O}_2^T = (0,T) \times \mathcal{O}_2$ and $\Sigma = (0,T) \times \Gamma$. We consider the following nonlinear heat equation:

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + f(y) = k\mathbf{1}_\omega + v_1\mathbf{1}_{\mathcal{O}_1} + v_2\mathbf{1}_{\mathcal{O}_2} & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0,.) = y^0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $y^0 \in L^2(\Omega)$ and the controls k, v_1 and v_2 belong to $L^2(\omega^T)$, $L^2(\mathcal{O}_1^T)$ and $L^2(\mathcal{O}_2^T)$, respectively. The function $\mathbf{1}_{\mathbb{X}}$ denotes the characteristic function of the set $\mathbb{X}$. We assume that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions:

$$\begin{cases} \exists K > 0 : \ |f(u_1) - f(u_2)| \leq K|u_1 - u_2|, \quad \forall\, u_1, u_2 \in \mathbb{R}, \\ f(0) = 0, \\ f \in \mathcal{C}^1(\mathbb{R}). \end{cases} \tag{2}$$

The first assumption of (2) means that f is a globally Lipschitz function. We also assume that the unbounded sets Ω and ω are such that

$$\Omega \setminus \omega \quad \text{is bounded.} \tag{3}$$

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If $y^0 \in L^2(\Omega)$, $k \in L^2(\omega^T)$, $v_i \in L^2(\mathcal{O}_i^T)$, $i = 1, 2$ and f satisfies (2), then, we prove that the system (1) admits a unique solution

$$y := y(k, v_1, v_2) \in L^2((0, T); H_0^1(\Omega)) \cap \mathcal{C}([0, T]; L^2(\Omega)).$$

Let $\mathcal{O}_{1,d}$, $\mathcal{O}_{2,d} \subset \Omega$ be two open subsets, representing the observation domains. We define the following functionals:

$$J_i(k; v_1, v_2) = \frac{\alpha_i}{2} \|y(k, v_1, v_2) - z_{i,d}\|^2_{L^2((0,T)\times\mathcal{O}_{i,d})} + \frac{N_i}{2} \|v_i\|^2_{L^2(\mathcal{O}_i^T)}, \quad i = 1, 2, \tag{4}$$

where α_i and N_i are positive constants and $z_{i,d} \in L^2((0, T) \times \mathcal{O}_{i,d})$, $i = 1, 2$ are desired states.

In this paper, we analyze the null controllability of system (1) following hierarchic control technique. More precisely, we apply the Stackelberg–Nash method, which combines the optimization technique of Stackelberg and noncooperative optimization technique of Nash. In order to explain the methodology, we consider the two following problems:

Problem 1.1: Let ω, $\mathcal{O}_1$ and $\mathcal{O}_2$ be three nonempty subsets of Ω. Given $k \in L^2(\omega^T)$ and $y^0 \in L^2(\Omega)$, find the controls $\hat{v}_1 := \hat{v}_1(k) \in L^2(\mathcal{O}_1^T)$ and $\hat{v}_2 := \hat{v}_2(k) \in L^2(\mathcal{O}_2^T)$ such that

$$J_1(k; \hat{v}_1, \hat{v}_2) = \min_{v_1 \in L^2(\mathcal{O}_1^T)} J_1(k; v_1, \hat{v}_2), \tag{5}$$

and

$$J_2(k; \hat{v}_1, \hat{v}_2) = \min_{v_2 \in L^2(\mathcal{O}_2^T)} J_2(k; \hat{v}_1, v_2), \tag{6}$$

where the functionals J_1 and J_2 are defined by (4).

Problem 1.2: Let ω, $\mathcal{O}_1$ and $\mathcal{O}_2$ be three nonempty unbounded subsets of Ω such that $\mathcal{O}_i \cap \omega = \emptyset$. Assume that (3) holds true. Let also $(\hat{v}_1, \hat{v}_2)$ be the solution obtained for Problem 1.1. Given $y^0 \in L^2(\Omega)$, find a control $k \in L^2(\omega^T)$ such that if $\hat{y} = y(t, x; k, \hat{v}_1(k), \hat{v}_2(k))$ is solution of (1), then

$$\hat{y}(T, x) = y(T, x; k, \hat{v}_1(k), \hat{v}_2(k)) = 0, \quad \text{for } x \in \Omega. \tag{7}$$

The motivation of the hierarchic control comes for example from environmental problem. The system (1) can be used to describe the diffusion of a pollutant (e.g. the chemical product) named y in a river Ω. Our goal here is to bring the concentration of this pollutant to zero at the final time $T > 0$ with appropriate control denoted k, trying meanwhile to keep the concentration of this pollutant to a desired state in $\mathcal{O}_{i,d}$ along the interval $(0, T)$ with another controls named v_i.

Remark 1.1: (1) Note that (5)–(6) are equivalent to

$$J_1(k; \hat{v}_1, \hat{v}_2) \le J_1(k; v_1, \hat{v}_2), \quad \forall\, v_1 \in L^2(\mathcal{O}_1^T),$$
$$J_2(k; \hat{v}_1, \hat{v}_2) \le J_2(k; \hat{v}_1, v_2), \quad \forall\, v_2 \in L^2(\mathcal{O}_2^T).$$

(2) Any pair $(\hat{v}_1, \hat{v}_2)$ satisfying (5)–(6) is called a Nash equilibrium for J_i, $i = 1, 2$ given by (4) and associated to k.

(3) If the functionals J_i, $i = 1, 2$ are convex, then $(\hat{v}_1, \hat{v}_2)$ is a Nash equilibrium for J_i, $i = 1, 2$ if and only if

$$\frac{\partial J_1}{\partial v_1}(k; \hat{v}_1, \hat{v}_2)(v_1, 0) = 0, \quad \forall\, v_1 \in L^2(\mathcal{O}_1^T), \quad \hat{v}_i \in L^2(\mathcal{O}_i^T) \tag{8}$$

and

$$\frac{\partial J_2}{\partial v_2}(k; \hat{v}_1, \hat{v}_2)(0, v_2) = 0, \quad \forall\, v_2 \in L^2(\mathcal{O}_1^T), \quad \hat{v}_i \in L^2(\mathcal{O}_i^T). \tag{9}$$

Problem 1.1 and Problem 1.2 constitute a hierarchic control strategy for model (1). The Stackelberg leadership model is a multiple-objective optimization approach initiated by H. von Stackelberg in [1]. This model is a strategic game in Economics in which two firms compete on the market with the same product. The first to act must integrate the reaction of the other company in the choices it makes in the amount of product that it decides to put on the market. There are some literature on the Stackelberg strategy in the framework of partial differential equations. In [2], J. L. Lions used the Stackelberg strategy for a linear parabolic equation with two controls named follower and leader. The follower aimed to bring the state of the system not too far from a desired state, while the leader has to steer the state at final time to a small neighborhood of a given state. Several other papers, see for instance [3–8], applied the Stackelberg control strategy to solve a wide variety of problems. Recently, in [9,10], the authors used the Stackelberg strategy to solve problems with incomplete or missing data.

There are also in the literature some results about Stackelberg–Nash strategy for partial differential equations (PDEs). J. I. Diaz and J. L. Lions in [11,12] studied the Stackelberg–Nash strategies for the approximate controllability of some parabolic equations with one leader and N followers. In [13], F. Guillén-Gonzàlez et al. studied the approximate controllability of Stackelberg–Nash strategy for Stokes equations with three controls. In [14], F. D Araruna et al. developed the first hierarchical results within the exact controllability framework for a linear and semi-linear parabolic equations. This strategy involved three controls: one leader and two followers. Each follower was supposed to bring the state of the system to a desired state while the leader solved an exact controllability problem. This previous work motivated some authors and a lot of other results appeared, see for instance, [15–20] to solve a variety of systems. Note that the above works on hierarchic control for partial differential equations were considered in bounded domains. In [21], the authors proved the Stackelberg strategy in the sense of null controllability for a linear parabolic equation in an unbounded domain. As far as we know, Stackelberg–Nash strategy has not yet been considered in an unbounded domain in the sense of null controllability. However in [22], I. P. de Jesus et al. established Stackelberg–Nash strategy for the linear heat equation in an unbounded domain $\Omega = \mathbb{R}^n$. They spent the case of approximate Stackelberg–Nash controllability.

In this paper we are concerned with Stackelberg–Nash null controllability problem in an unbounded domain involving three controls: two followers and one leader. The first problem is a Nash equilibrium which is a noncooperative optimization approach introduced in [23] by J. F. Nash. Assuming that the leader has made his choice of policy, the objective of each follower in Problem 1.1 is to move the state of the system not too far to his desired state. The leader solves a null controllability problem in an unbounded domain.

Null controllability problem of parabolic equations in unbounded domains has been studied by some authors. The first positive result was obtained by S. B. De Menezes et al. in [24]. The authors showed the null controllability of a semilinear heat equation in an unbounded domain with nonlinearities of the form $f(y)$, the real function f being of class $\mathcal{C}^1$ and globally Lipschitz. The results were achieved by means of a fixed-point theorem and under the assumption that the uncontrolled domain is bounded. In [25], V. R. Cabanillas et al. extended the previous result for a nonlinear parabolic equation with the nonlinearities of the form $f(y, \nabla y)$. Under the assumption that the uncontrolled region is bounded, M. G. Burgos et. al. [26] proved the null controllability of a semi-linear heat equation with the nonlinearities of the form $f(y, \nabla y)$ which grows slower than $|y| \log^{3/2}(1 + |y| + |\nabla y|) + |\nabla y| \log^{1/2}(1 + |y| + |\nabla y|)$ at infinity.

Following the ideas of the above papers on the null controllability problem in unbounded domains, we study the Stackelberg–Nash null controllability of a nonlinear heat equation in an unbounded domain. Using the fact that the control which bring the system to rest at final time acts on an open unbounded set such that the uncontrolled domain is bounded, we prove using appropriate Carleman inequalities and fixed-point theorem that the system (1) is Stackelberg–Nash null controllable. The novelty of this paper is that, we extend Stackelberg–Nash strategy to an unbounded domain.

1.1. Main results

To state the main contributions of this paper, we will have to impose the following assumptions:

$$\begin{cases} \mathcal{O}_{1,d} = \mathcal{O}_{2,d} : \text{the common observability set will be denoted by } \mathcal{O}_d, \\ \mathcal{O}_d \cap \omega \neq \emptyset. \end{cases} \tag{10}$$

If $f(y) = ay$ and we assume that $a \in L^\infty(Q)$, then the system (1) is linear and we have the following result.

Theorem 1.1: *Suppose that* (10) *holds and that* $N_i,\ i = 1,2$ *are large enough. Then, there exist two positive real weight functions* $\theta = \theta(t)$ *and* $\varpi = \varpi(t,x)$ *(the definition of* θ *and* ϖ *will be given later) such that for any* $z_{i,d} \in L^2(\mathcal{O}_d^T)$ *satisfying*

$$\int_0^T \int_{\mathcal{O}_d} \theta^{-2} |z_{i,d}|^2 \, \mathrm{d}x \, \mathrm{d}t < +\infty, \quad i = 1,2, \tag{11}$$

and for any $y^0 \in L^2(\Omega)$, *there exists a unique control* $\hat{h} \in L^2(\omega^T)$ *and the corresponding Nash equilibrium* $(\hat{v}_1, \hat{v}_2)$ *such that the solution of* (1) *satisfies* (7). *Moreover*

$$\hat{h} = \hat{\rho} \quad \text{in } \omega_T, \tag{12}$$

where $\hat{\rho}$ *satisfies*

$$\begin{cases} -\dfrac{\partial \hat{\rho}}{\partial t} - \Delta \hat{\rho} + a\hat{\rho} = \left(\alpha_1 \hat{\Psi}_1 + \alpha_2 \hat{\Psi}_2\right) \mathbf{1}_{\mathcal{O}_d} & \text{in } Q, \\ \hat{\rho} = 0 & \text{on } \Sigma, \end{cases} \tag{13}$$

and $\hat{\Psi}_i,\ i = 1,2$ *are solutions of*

$$\begin{cases} \dfrac{\partial \hat{\Psi}_i}{\partial t} - \Delta \hat{\Psi}_i + a\hat{\Psi}_i = -\dfrac{1}{N_i} \hat{\rho}_i \mathbf{1}_{\mathcal{O}_i} & \text{in } Q, \\ \hat{\Psi}_i = 0 & \text{on } \Sigma, \\ \hat{\Psi}_i(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{14}$$

In addition, there exists a constant $C = C(\|a\|_{L^\infty(Q)}, T, \tau_0, \tau_1, d_0) > 0$ *such that*

$$\|\hat{h}\|_{L^2(\omega^T)} \leq C \left(\sum_{i=1}^{2} \left\| \frac{1}{\theta} z_{i,d} \right\|_{L^2(\mathcal{O}_d^T)} + \|y^0\|_{L^2(\Omega)} \right). \tag{15}$$

In the semi-linear case, we do not have the convexity of the functionals $J_i,\ i = 1,2$ in general and this motivates the following weaker definition.

Definition 1.1: Let k be given. The pair $(\hat{v}_1, \hat{v}_2)$ is called a Nash quasi-equilibrium for the functionals $J_i,\ i = 1,2$ associated to k if the conditions (8) and (9) are satisfied.

The following result holds in the semi-linear case.

Theorem 1.2: *Suppose that* (2), (3) *and* (10) *hold and* $f \in W^{1,\infty}(\mathbb{R})$. *Then, there exist two positive real weight functions* $\theta = \theta(t)$ *and* $\varpi = \varpi(t,x)$ *such that if (11) holds, for any* $y^0 \in L^2(\Omega)$, *there exists a control* $\hat{k} \in L^2(\omega^T)$ *and associated Nash quasi-equilibrium* $(\hat{v}_1, \hat{v}_2)$ *such that the solution of* (1) *satisfies* (7).

In the semi-linear case, there are some situations where the concepts of Nash equilibrium and Nash quasi-equilibrium are equivalents. An answer is given by the following result:

Proposition 1.1: *Assume that $f \in W^{2,\infty}(\mathbb{R})$ and $z_{i,d} \in L^\infty(\mathcal{O}_d^T)$ for $i = 1, 2$. Suppose that $y^0 \in L^2(\Omega)$ and $N \leq 12$. Then, there exists a positive constant C independent of N_i, $i = 1, 2$ such that, if $k \in L^2(\omega^T)$ and the N_i are large enough, the pair $(\hat{v}_1, \hat{v}_2)$ is a Nash equilibrium for J_i, $i = 1, 2$ of (1).*

Remark 1.2:
- In this paper, we assume that $\mathcal{O}_i \cap \omega = \emptyset$. This means that the domain of followers control and the leader control are disjoints. Note that, in a realistic situation, the leader control cannot decide what to do at the points in the domain of followers. Indeed, if $\mathcal{O}_i \cap \omega \neq \emptyset$, once the leader has chosen it strategy, the followers can modify the leader's strategy at those points.
- The assumption (3) is very important. Indeed, the main difficulty with unbounded domains is that we lose the compactness of the Sobolev embedding. So to overcome this difficulty, we will need this hypothesis and then the nonlinear system is reduced to the case where the nonlinearity is now supported in a bounded domain.
- The first assumption in (10) will be used in Section 2.2 to obtain the observability inequality of Carleman type very important tool to solve controllability problem.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1 corresponding to the Stackelberg–Nash null controllability for a linear auxiliary system. In Section 3, we prove Theorem 1.2 by using a fixed point argument. A conclusion is given in Section 4.

2. The linear case

The purpose of this section is to prove Theorem 1.1. We will do this in the next three subsections. Now, we are concerned with the Stackelberg–Nash null controllability for the following linear system

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + ay = h\mathbf{1}_\omega + v_1\mathbf{1}_{\mathcal{O}_1} + v_2\mathbf{1}_{\mathcal{O}_2} & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0,.) = y^0 & \text{in } \Omega, \end{cases} \tag{16}$$

where $h \in L^2(\omega^T)$, $v_1 \in L^2(\mathcal{O}_1^T)$, $v_2 \in L^2(\mathcal{O}_2^T)$ and $y^0 \in L^2(\Omega)$. This means that we want to solve Problem 1.1 and Problem 1.2 for system (16).

We assume in this section that the potential $a = a(t, x)$ is in $L^\infty(Q)$. From now on, $\|\cdot\|_\infty$ will denote the norm in $L^\infty(Q)$ and we will write $C(X)$ to denote a positive constant whose value varies from a line to line but depends on X.

Under the assumptions on the data, system (16) has a unique solution $y(t, x) := y(h, v_1, v_2) = y(t, x; h, v_1, v_2) \in L^2((0, T); H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega))$. Moreover, we have that there exists a positive constant $C = C(\|a\|_\infty, T)$ such that

$$\|y(T,.)\|^2_{L^2(\Omega)} + \|y\|^2_{L^2((0,T);H_0^1(\Omega))} \leq C\left(\|y^0\|^2_{L^2(\Omega)} + \|h\|^2_{L^2(\omega^T)}\right) + C\left(\|v_1\|^2_{L^2(\mathcal{O}_1^T)} + \|v_2\|^2_{L^2(\mathcal{O}_2^T)}\right). \tag{17}$$

Actually,

$$C(\|a\|_\infty, T) = e^{2(2\|a\|_\infty+1)T}. \tag{18}$$

2.1. *Problem* 1.1 *for system* (16)

Let $\mathcal{H}$ be the Hilbert space defined by:

$$\mathcal{H} = L^2(\mathcal{O}_1^T) \times L^2(\mathcal{O}_2^T), \tag{19}$$

with the scalar product:

$$((f,\varphi),(g,\phi))_{\mathcal{H}} = \int_{\mathcal{O}_1^T} f\, g \;\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{O}_2^T} \varphi\, \phi \;\mathrm{d}x\,\mathrm{d}t, \quad \text{for all } (f,\varphi),(g,\phi) \in \mathcal{H}. \tag{20}$$

We are interested in Problem 1.1 for the linear system (16), that is, for any $h \in L^2(\omega^T)$, find the controls $\hat{v}_1 := \hat{v}_1(h) \in L^2(\mathcal{O}_1^T)$ and $\hat{v}_2 := \hat{v}_2(h) \in L^2(\mathcal{O}_2^T)$ such that

$$J_1(h;\hat{v}_1,\hat{v}_2) = \min_{v_1 \in L^2(\mathcal{O}_1^T)} J_1(h;v_1,\hat{v}_2) \tag{21}$$

and

$$J_2(h;\hat{v}_1,\hat{v}_2) = \min_{v_2 \in L^2(\mathcal{O}_2^T)} J_2(h;\hat{v}_1,v_2), \tag{22}$$

where for $i = 1, 2$,

$$J_i(h;v_1,v_2) = \frac{\alpha_i}{2}\|y(h,v_1,v_2) - z_{i,d}\|^2_{L^2((0,T)\times\mathcal{O}_{i,d})} + \frac{N_i}{2}\|v_i\|^2_{L^2(\mathcal{O}_i^T)}, \tag{23}$$

with $\alpha_i, N_i > 0$, $z_{i,d} \in L^2((0,T)\times\mathcal{O}_{i,d}))$, $i = 1, 2$ and $y(h,v_1,v_2)$ being the solution of the linear system (16).

Since the system (16) is linear, then the functionals J_i, $i = 1,2$ given by (23) are convex. Then using Remark 1.1, we have that $(\hat{v}_1,\hat{v}_2)$ is solution of (21)–(22) if and only if

$$\frac{\partial J_1}{\partial v_1}(h;\hat{v}_1,\hat{v}_2)(v_1,0) = 0, \quad \forall\, v_1 \in L^2(\mathcal{O}_1^T), \quad \hat{v}_i \in L^2(\mathcal{O}_i^T) \tag{24}$$

and

$$\frac{\partial J_2}{\partial v_2}(h;\hat{v}_1,\hat{v}_2)(0,v_2) = 0, \quad \forall\, v_2 \in L^2(\mathcal{O}_1^T), \quad \hat{v}_i \in L^2(\mathcal{O}_i^T). \tag{25}$$

2.1.1. *Existence and uniqueness of a Nash equilibrium*

We have the following result:

Proposition 2.1: *Let $h \in L^2(\omega^T)$. Assume that*

$$N_1 > \mathrm{e}^{2(2\|a\|_\infty+1)T}\frac{\alpha_2}{4} \quad \text{and} \quad N_2 > \mathrm{e}^{2(2\|a\|_\infty+1)T}\frac{\alpha_1}{4}. \tag{26}$$

Then for any $h \in L^2(\omega^T)$, there exists a unique Nash equilibrium $(\hat{v}_1,\hat{v}_2) \in \mathcal{H}$ for J_1 and J_2 associated to h. Moreover, there exists a constant $C = C(\|a\|_\infty, T, \alpha_1, \alpha_2)$ such that

$$\|(\hat{v}_1,\hat{v}_2)\|_{\mathcal{H}} \le \frac{1}{\gamma}C\left(\sum_{i=1}^{2}\|z_{i,d}\|_{L^2((0,T)\times\mathcal{O}_{i,d})} + \|h\|_{L^2(\omega^T)} + \|y^0\|_{L^2(\Omega)}\right), \tag{27}$$

where

$$\gamma = \min\left(N_1 - \mathrm{e}^{2(1+2\|a\|_\infty)T}\frac{\alpha_2}{4},\; N_2 - \mathrm{e}^{2(1+2\|a\|_\infty)T}\frac{\alpha_1}{4}\right) > 0. \tag{28}$$

Proof: We proceed as in [12,14]. We define by L_i the linear and continuous operator from $L^2(\mathcal{O}_i^T)$ to $L^2((0,T);H_0^1(\Omega))$ such that $L_i v_i = z_i$ where $z_i,\ i=1,2$ is the solution to the following system

$$\begin{cases} \dfrac{\partial z_i}{\partial t} - \Delta z_i + a z_i = v_i \mathbf{1}_{\mathcal{O}_i} & \text{in } Q, \\ z_i = 0 & \text{on } \Sigma, \\ z_i(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{29}$$

Then $z_i \in L^2((0,T);H_0^1(\Omega))$ is unique and there exists a constant $C(\|a\|_\infty, T) > 0$ such that

$$\|L_i v_i\|_{L^2((0,T);H_0^1(\Omega))} \le C\|v_i\|_{L^2(\mathcal{O}_i^T)}. \tag{30}$$

We have that $(\hat{v}_1,\hat{v}_2)$ is a Nash equilibrium for (J_1,J_2) given by (23) if and only if conditions (24)–(25) are satisfied. This means that the following relation holds true:

$$\alpha_i \int_Q (\hat{y} - z_{i,d}) z_i \ \mathrm{d}x\,\mathrm{d}t + N_i \int_{\mathcal{O}_i^T} \hat{v}_i v_i \ \mathrm{d}x\,\mathrm{d}t = 0, \quad \text{for all } v_i \in L^2(\mathcal{O}_i^T), \tag{31}$$

where $\hat{y} = y(h,\hat{v}_1,\hat{v}_2)$ and $z_i,\ i=1,2$ satisfies (29).

According to the definition of L_i, any solution y to (16) can be decomposed as $\hat{y} = L_1\hat{v}_1 + L_2\hat{v}_2 + l$, where $l \in L^2((0,T);H_0^1(\Omega))$ satisfies

$$\begin{cases} \dfrac{\partial l}{\partial t} - \Delta l + a l = h\mathbf{1}_\omega & \text{in } Q, \\ l = 0 & \text{on } \Sigma, \\ l(0,.) = y^0 & \text{in } \Omega. \end{cases} \tag{32}$$

Let L_i^* be the adjoint of operator L_i. Then L_i^* is a linear and continuous operator from $L^2((0,T);H_0^1(\Omega))$ to $L^2(\mathcal{O}_i^T)$. If we replace $\hat{y}$ in (31) by $L_1\hat{v}_1 + L_2\hat{v}_2 + l$, then (31) becomes

$$\int_{\mathcal{O}_i^T} \left[\alpha_i L_i^*(L_1\hat{v}_1 + L_2\hat{v}_2 - (z_{i,d} - l))v_i + N_i\hat{v}_i\right] v_i \ \mathrm{d}x\,\mathrm{d}t = 0, \quad \text{for all } v_i \in L^2(\mathcal{O}_i^T).$$

Thus, $(\hat{v}_1,\hat{v}_2)$ is a Nash equilibrium for (J_1,J_2) given by (23) if and only if

$$\alpha_i L_i^*(L_1\hat{v}_1 + L_2\hat{v}_2) + N_i\hat{v}_i = \alpha_i L_i^*(z_{i,d} - l) \quad \text{in } L^2(\mathcal{O}_i^T), \quad i = 1,2. \tag{33}$$

Now, we define the operator $\mathbb{T} : \mathcal{H} \to \mathcal{H}$ by

$$\mathbb{T}(\hat{v}_1,\hat{v}_2) = \left(\alpha_1 L_1^*(L_1\hat{v}_1 + L_2\hat{v}_2) + N_1\hat{v}_1, \alpha_2 L_2^*(L_1\hat{v}_1 + L_2\hat{v}_2) + N_2\hat{v}_2\right)$$

and we introduce the bilinear functional $\mathcal{B} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ defined by

$$\mathcal{B}\left((\hat{v}_1,\hat{v}_2),(v_1,v_2)\right) := \left(\mathbb{T}(\hat{v}_1,\hat{v}_2),(v_1,v_2)\right)_{\mathcal{H}},$$

where $(.,.)_{\mathcal{H}}$ denotes the scalar product on $\mathcal{H}$ given by (20).

Set

$$\Lambda = \left(\alpha_1 L_1^*(z_{1,d} - l), \alpha_2 L_2^*(z_{2,d} - l)\right).$$

Then it follows from (33) that the existence and uniqueness of the Nash equilibrium for (J_1,J_2) is reduced to the existence and uniqueness of solution of the following problem: let $h \in L^2(\omega^T)$, find

$(\hat{v}_1, \hat{v}_2) \in \mathcal{H}$ such that

$$\mathcal{B}\left((\hat{v}_1, \hat{v}_2), (v_1, v_2)\right) = (\Lambda, (v_1, v_2))_{\mathcal{H}}, \quad \text{for all } (v_1, v_2) \in \mathcal{H}. \tag{34}$$

Observing on the one hand that (30) holds for $v_i = \hat{v}_i$, and on the other hand that,

$$\|l\|_{L^2((0,T);H_0^1(\Omega))} \leq C(\|a\|_\infty, T)\left(\|h\|_{L^2(\omega^T)} + \|y^0\|_{L^2(\Omega)}\right), \tag{35}$$

because l is solution of (32), we prove using Cauchy–Schwarz inequality that

$$\left|\mathcal{B}\left((\hat{v}_1, \hat{v}_2), (v_1, v_2)\right)\right| \leq C\|(\hat{v}_1, \hat{v}_2)\|_{\mathcal{H}}\|(v_1, v_2)\|_{\mathcal{H}}, \tag{36}$$

where $C = C(\|a\|_\infty, T, \alpha_1, \alpha_2, N_1, N_2) > 0$.

Using again (30) and Young's inequality, we prove that if $N_1 > e^{2(1+2\|a\|_\infty)T}\frac{\alpha_2}{4}$ and $N_2 > e^{2(1+2\|a\|_\infty)T}\frac{\alpha_1}{4}$, then $\mathcal{B}$ is coercive, that is,

$$|\mathcal{B}((v_1, v_2), (v_1, v_2))| \geq \gamma\|(v_1, v_2)\|_{\mathcal{H}}^2, \tag{37}$$

where γ is defined by (28).

Now using Cauchy–Schwarz inequality, (30) and (35), we have that

$$(\Lambda, (v_1, v_2))_{\mathcal{H}} \leq C\|(v_1, v_2)\|_{\mathcal{H}}, \tag{38}$$

where $C = C(\|a\|_\infty, T, \alpha_1, \alpha_2)(\sum_{i=1}^2 \|z_{i,d}\|^2_{L^2((0,T)\times\mathcal{O}_{i,d})} + \|h\|^2_{L^2(\omega^T)} + \|y^0\|^2_{L^2(\Omega)})^{1/2} > 0$.

Finally, (36), (37) and (38) proves that, the bilinear functional $\mathcal{B}$ is continuous on $\mathcal{H} \times \mathcal{H}$, coercive on $\mathcal{H}$ and that the linear functional $(v_1, v_2) \mapsto (\Lambda, (v_1, v_2))_{\mathcal{H}}$ is continuous on $\mathcal{H}$. Therefore, the Lax–Milgram's theorem allows us to say that there exists a unique Nash equilibrium $(\hat{v}_1, \hat{v}_2) \in \mathcal{H}$.

Now taking $(v_1, v_2) = (\hat{v}_1, \hat{v}_2)$ in (34) and using (37) and (38), we deduce that

$$\|(\hat{v}_1, \hat{v}_2)\|_{\mathcal{H}} \leq \frac{1}{\gamma}C\left(\sum_{i=1}^2 \|z_{i,d}\|^2_{L^2((0,T)\times\mathcal{O}_{i,d})} + \|h\|^2_{L^2(\omega^T)} + \|y^0\|^2_{L^2(\Omega)}\right)^{1/2},$$

where $C = C(\|a\|_\infty, T, \alpha_1, \alpha_2) > 0$ and γ is given by (28). Hence, we obtain (27). ■

2.1.2. Optimality system for a Nash equilibrium

In order to give the optimality system that characterizes the Nash equilibrium $(\hat{v}_1, \hat{v}_2)$ for the cost functionals (J_1, J_2), we interpret relation (31). So, we consider the adjoint state $\hat{p}_i$, $i = 1, 2$ solution of

$$\begin{cases} -\dfrac{\partial \hat{p}_i}{\partial t} - \Delta\hat{p}_i + a\hat{p}_i = \alpha_i(\hat{y} - z_{i,d})\mathbf{1}_{\mathcal{O}_{i,d}} & \text{in } Q, \\ \hat{p}_i = 0 & \text{on } \Sigma, \\ \hat{p}_i(T, .) = 0 & \text{in } \Omega. \end{cases} \tag{39}$$

Since $\alpha_i(\hat{y} - z_{i,d})\mathbf{1}_{\mathcal{O}_{i,d}} \in L^2(Q)$, we deduce that the system (39) has a unique solution in $\hat{p}_i \in L^2((0,T); H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega))$, $i = 1, 2$. Thus, if we multiply (29) by $\hat{p}_i$, that is the solution of (39) and integrate by parts over Q, we obtain that

$$\alpha_i \int_Q z_i(\hat{y} - z_{i,d})\mathbf{1}_{\mathcal{O}_{i,d}} \, \mathrm{d}x\,\mathrm{d}t = \int_{\mathcal{O}_i^T} v_i \hat{p}_i \, \mathrm{d}x\,\mathrm{d}t = 0, \quad \text{for every } v_i \in L^2(\mathcal{O}_i^T).$$

This, together with (31) gives

$$\int_{\mathcal{O}_i^T} (\hat{p}_i + N_i \hat{v}_i) v_i \, \mathrm{d}x\,\mathrm{d}t = 0, \quad \text{for every } v_i \in L^2(\mathcal{O}_i^T).$$

This means that,

$$\hat{v}_i = -\frac{1}{N_i}\hat{p}_i \quad \text{in } \mathcal{O}_i^T, \quad i = 1, 2.$$

We have proved the following results.

Proposition 2.2: *Let $h \in L^2(\omega^T)$. Assume that* (26) *holds true. Then, the pair $(\hat{v}_1, \hat{v}_2)$ is a Nash equilibrium for* (21)–(22) *if and only if*

$$\hat{v}_i = -\frac{1}{N_i}\hat{p}_i \quad \text{in } \mathcal{O}_i^T, \quad i = 1, 2, \tag{40}$$

where $(\hat{y}, \hat{p}_i)$ is solution of the following systems

$$\begin{cases} \dfrac{\partial \hat{y}}{\partial t} - \Delta \hat{y} + a\hat{y} = h\mathbf{1}_\omega - \dfrac{1}{N_1}\hat{p}_1 \mathbf{1}_{\mathcal{O}_1} - \dfrac{1}{N_2}\hat{p}_2 \mathbf{1}_{\mathcal{O}_2} & \text{in } Q, \\ \hat{y} = 0 & \text{on } \Sigma, \\ \hat{y}(0,.) = y^0 & \text{in } \Omega \end{cases} \tag{41}$$

and

$$\begin{cases} -\dfrac{\partial \hat{p}_i}{\partial t} - \Delta \hat{p}_i + a\hat{p}_i = \alpha_i(\hat{y} - z_{i,d})\mathbf{1}_{\mathcal{O}_{i,d}} & \text{in } Q, \\ \hat{p}_i = 0 & \text{on } \Sigma, \\ \hat{p}_i(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{42}$$

2.2. Carleman inequalities

In this subsection, we establish the inequality of observability useful for solving the null controllability of the cascade linear system (41)–(42). We assume that ω_1 is an unbounded set with $\omega_1 \subset \omega$ and such that

$$\Omega \setminus \omega_1 \quad \text{is bounded.} \tag{43}$$

Remark 2.1: Note that with (43), we have that assumption (3) holds true.

Since the second assumption of (10) holds, that means $\mathcal{O}_d \cap \omega \neq \emptyset$, then there exist an open set ω_0 such that

$$\omega_0 \subset \omega_1 \subset \mathcal{O}_d \cap \omega \quad \text{with } d_0 = dist\,(\omega_0, \Omega \setminus \bar{\omega}_1) > 0, \tag{44}$$

and a function ψ such that

$$\begin{cases} \psi \in \mathcal{C}^2(\bar{\Omega}), \quad \psi \geq 0 & \text{in } \Omega, \\ |\nabla \psi| \geq \tau_0 > 0 & \text{in } \bar{\Omega} \setminus \omega_0, \\ \dfrac{\partial \psi}{\partial \nu} \leq 0 & \text{on } \partial\Omega, \quad \sum_{|\beta| \leq 2} |D^\beta \psi| \leq \tau_1 \quad \text{in } \Omega, \end{cases} \tag{45}$$

where τ_0 and τ_1 are two positive constants. For the construction of such a function ψ in the case when the domain Ω is unbounded, we refer to [26].

Let λ be a positive real number. For any $(t,x) \in Q$, we define the functions

$$\varphi(t,x) = \frac{e^{\lambda(\psi(x)+m_1)}}{t(T-t)}, \tag{46}$$

$$\eta(t,x) = \frac{e^{\lambda(\|\psi\|_{L^\infty(\Omega)}+m_2)} - e^{\lambda(\psi(x)+m_1)}}{t(T-t)}, \tag{47}$$

with $m_2 > m_1$. Then, there exists a positive constant $C(T)$ such that,

$$\left|\frac{\partial \eta}{\partial t}\right| \leq C(T)\varphi^2, \tag{48a}$$

$$\left|\frac{\partial \varphi}{\partial t}\right| \leq C(T)\varphi^2. \tag{48b}$$

For any $F_0 \in L^2(Q)$ and $z_0 \in L^2(\Omega)$, we consider the following system:

$$\begin{cases} -\dfrac{\partial z}{\partial t} - \Delta z = F_0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T,\cdot) = z_0 & \text{in } \Omega. \end{cases} \tag{49}$$

We have the following result which give a Carleman inequality for solution to the system (49).

Proposition 2.3 ([26]): *Suppose that assumptions* (44)–(45) *hold true. Let φ and η be defined by* (46) *and* (47)*, respectively. Then, there exist positive constants $\sigma_1(\tau_0,\tau_1,d_0) \geq 1$, $\lambda_1(\tau_0,\tau_1,d_0) \geq 1$ and $C(\tau_0,\tau_1,d_0) > 0$ such that for all $\lambda \geq \lambda_1$, $s \geq s_1 = \sigma_1(T+T^2)$, and for any solution of* (49) *denoted by z, we have*

$$\begin{aligned} &s\lambda^2 \int_Q e^{-2s\eta}\varphi|\nabla z|^2 \,dx\,dt + s^3\lambda^4 \int_Q e^{-2s\eta}\varphi^3|z|^2 \,dx\,dt \\ &\quad \leq C(\tau_0,\tau_1,d_0)\left(s^3\lambda^4 \int_0^T \int_{\omega_1} e^{-2s\eta}\varphi^3|z|^2 \,dx\,dt + \int_Q e^{-2s\eta}|F_0|^2 \,dx\,dt\right). \end{aligned} \tag{50}$$

Now, consider the following system:

$$\begin{cases} -\dfrac{\partial z}{\partial t} - \Delta z + az = f & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T,\cdot) = z_0 & \text{in } \Omega, \end{cases} \tag{51}$$

with $f \in L^2(Q)$ and $z_0 \in L^2(\Omega)$. Then, we have the following result for (51).

Proposition 2.4: *Under the assumptions of Proposition* 2.3*, there exist positive constants $\sigma_1(\tau_0,\tau_1,d_0) \geq 1$, $\lambda_1(\tau_0,\tau_1,d_0) \geq 1$, $s_2 = \max(s_1, 4C(\tau_0,\tau_1,d_0)\|a\|_\infty^2) \geq 1$ and $C = C(\tau_0,\tau_1,d_0) > 0$ such that for all $\lambda \geq \lambda_1$, $s \geq s_2$, and for any z solution of* (51)*, we have*

$$\begin{aligned} &s\lambda^2 \int_Q e^{-2s\eta}\varphi|\nabla z|^2 \,dx\,dt + s^3\lambda^4 \int_Q e^{-2s\eta}\varphi^3|z|^2 \,dx\,dt \\ &\quad \leq C\left(s^3\lambda^4 \int_0^T \int_{\omega_1} e^{-2s\eta}\varphi^3|z|^2 \,dx\,dt + \int_Q e^{-2s\eta}|f|^2 \,dx\,dt\right). \end{aligned} \tag{52}$$

Proof: We write system (51) as

$$\begin{cases} -\dfrac{\partial z}{\partial t} - \Delta z = (-az + f) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T,\cdot) = z_0 & \text{in } \Omega. \end{cases}$$

Hence, z verifies (49) with $F_0 = -az + f \in L^2(Q)$. Thus, we can apply Proposition 2.3 to z and we deduce that there exists $C = C(\tau_0, \tau_1, d_0) > 0$ such that

$$\begin{aligned} & s\lambda^2 \int_Q e^{-2s\eta}\varphi|\nabla z|^2 \,dx\,dt + s^3\lambda^4 \int_Q e^{-2s\eta}\varphi^3|z|^2 \,dx\,dt \\ & \leq C\left(s^3\lambda^4 \int_0^T \int_{\omega_1} e^{-2s\eta}\varphi^3|z|^2 \,dx\,dt + \int_Q e^{-2s\eta}|f|^2 \,dx\,dt \right) \\ & \quad + \|a\|_\infty^2 C\left(\int_Q e^{-2s\eta}|z|^2 \,dx\,dt + s^2\lambda^2 \int_Q e^{-2s\eta}\varphi^2|z|^2 \,dx\,dt \right). \end{aligned}$$

Observing that $s,\ \lambda > 1$ and $\varphi^{-1} \in L^\infty(Q)$, it follows from the latter inequality that, there exists $C = C(\tau_0, \tau_1, d_0) > 0$ such that

$$\begin{aligned} & s\lambda^2 \int_Q e^{-2s\eta}\varphi|\nabla z|^2 \,dx\,dt + s^3\lambda^4 \int_Q e^{-2s\eta}\varphi^3|z|^2 \,dx\,dt \\ & \leq C\left(s^3\lambda^4 \int_0^T \int_{\omega_1} e^{-2s\eta}\varphi^3|z|^2 \,dx\,dt + \int_Q e^{-2s\eta}|f|^2 \,dx\,dt \right) \\ & \quad + \|a\|_\infty^2 C\left(2s^2\lambda^4 \int_Q e^{-2s\eta}\varphi^3|z|^2 \,dx\,dt \right). \end{aligned}$$

Choosing $s \geq s_2 = \max(s_1, 4C(\tau_0, \tau_1, d_0)\|a\|_\infty^2)$ in this latter inequality, we obtain (52). ∎

Remark 2.2: If we make a change of variable t for $T-t$ in (51), we have

$$\begin{cases} \dfrac{\partial \tilde{z}}{\partial t} - \Delta \tilde{z} + a\tilde{z} = \tilde{f} & \text{in } Q, \\ \tilde{z} = 0 & \text{on } \Sigma, \\ \tilde{z}(0,\cdot) = z_0 & \text{in } \Omega, \end{cases} \tag{53}$$

where $\tilde{z}(t,x) = z(T-t,x)$ and $\tilde{f}(t,x) = f(T-t,x)$. Then, the global Carleman inequality (52) is also valid for any $\tilde{z}$ solution of (53).

From now on, we will adopt for a suitable function z, the following notation

$$\mathcal{K}(z) = s\lambda^2 \int_Q e^{-2s\eta}\varphi|\nabla z|^2 \,dx\,dt + s^3\lambda^4 \int_Q e^{-2s\eta}\varphi^3|z|^2 \,dx\,dt. \tag{54}$$

For $\rho^T \in L^2(\Omega)$, we consider following systems:

$$\begin{cases} -\dfrac{\partial \rho}{\partial t} - \Delta\rho + a\rho = \alpha_1\Psi_1\mathbf{1}_{\mathcal{O}_{1,d}} + \alpha_2\Psi_2\mathbf{1}_{\mathcal{O}_{2,d}} & \text{in } Q, \\ \rho = 0 & \text{on } \Sigma, \\ \rho(T,\cdot) = \rho^T & \text{in } \Omega, \end{cases} \tag{55}$$

and for $i = 1, 2$,

$$\begin{cases} \dfrac{\partial \Psi_i}{\partial t} - \Delta \Psi_i + a\Psi_i = -\dfrac{1}{N_i}\rho \mathbf{1}_{\mathcal{O}_i} & \text{in } Q, \\ \Psi_i = 0 & \text{on } \Sigma, \\ \Psi_i(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{56}$$

Since the first assumption of (10) holds, if we set $\phi = \alpha_1 \Psi_1 + \alpha_2 \Psi_2$, then in view of (56), ϕ is solution to

$$\begin{cases} \dfrac{\partial \phi}{\partial t} - \Delta \phi + a\phi = -\dfrac{\alpha_1}{N_1}\rho \mathbf{1}_{\mathcal{O}_1} - \dfrac{\alpha_2}{N_2}\rho \mathbf{1}_{\mathcal{O}_2} & \text{in } Q, \\ \phi = 0 & \text{on } \Sigma, \\ \phi(0,.) = 0 & \text{in } \Omega, \end{cases} \tag{57}$$

and (55) can be rewritten as

$$\begin{cases} -\dfrac{\partial \rho}{\partial t} - \Delta \rho + a\rho = \phi \mathbf{1}_{\mathcal{O}_d} & \text{in } Q, \\ \rho = 0 & \text{on } \Sigma, \\ \rho(T,\cdot) = \rho^T & \text{in } \Omega. \end{cases} \tag{58}$$

In the following result, we present the Carleman inequality for solutions to systems (57)–(58).

Proposition 2.5: *Assume that the N_i, $(i = 1, 2)$ are large enough. Then, under the assumptions of Proposition 2.4, there exist positive constants $s_3 \geq 1$, $\lambda_2 \geq 1$ and $C = C(\tau_0, \tau_1, d_0, T) > 0$ such that for all $\lambda \geq \lambda_2$, $s \geq s_3$, the following estimate holds true for any solution (ϕ, ρ) of (57)–(58):*

$$\mathcal{K}(\rho) + \mathcal{K}(\phi) \leq Cs^7\lambda^9 \int_{\omega^T} \mathrm{e}^{-2s\eta}\varphi^7 |\rho|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{59}$$

Here, $s_3 = \max(s_2, 2C(\tau_0, \tau_1, d_0, N_1, N_2, \alpha_1, \alpha_2))$ and $\lambda_2 = \max(\lambda_1, 2C(\tau_0, \tau_1, d_0))$ with λ_1 and s_2 defined as in Proposition 2.4.

Proof: We proceed in two steps.

Step 1. We prove that there exist $s_3 \geq 1$, $\lambda_2 \geq 1$ and $C = C(\tau_0, \tau_1, d_0) > 0$ such that for any $s \geq s_3$ and $\lambda \geq \lambda_2$,

$$\mathcal{K}(\rho) + \mathcal{K}(\phi) \leq C\left(s^3\lambda^4 \int_0^T \int_{\omega_1} \mathrm{e}^{-2s\eta}\varphi^3 \left(|\rho|^2 + |\phi|^2\right) \,\mathrm{d}x\,\mathrm{d}t\right). \tag{60}$$

Applying (52) to the solution ρ of (58) and to the solution ϕ of (57) because of Remark 2.2, then using the notation (54), we, respectively, have that there exists $C = C(\tau_0, \tau_1, d_0) > 0$ such that

$$\begin{aligned} \mathcal{K}(\rho) &\leq C(\tau_0, \tau_1, d_0)\left(s^3\lambda^4 \int_0^T \int_{\omega_1} \mathrm{e}^{-2s\eta}\varphi^3|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_Q \mathrm{e}^{-2s\eta}|\phi|^2 \,\mathrm{d}x\,\mathrm{d}t\right) \\ &\leq C(\tau_0, \tau_1, d_0)\left(s^3\lambda^4 \int_0^T \int_{\omega_1} \mathrm{e}^{-2s\eta}\varphi^3|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_Q s^3\lambda^3\varphi^3\mathrm{e}^{-2s\eta}|\phi|^2 \,\mathrm{d}x\,\mathrm{d}t\right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}(\phi) \leq{}& C(\tau_0, \tau_1, d_0)s^3\lambda^4 \int_0^T \int_{\omega_1} \mathrm{e}^{-2s\eta}\varphi^3|\phi|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ C(\tau_0, \tau_1, d_0)\int_Q \mathrm{e}^{-2s\eta}\left|\frac{\alpha_1}{N_1}\rho\mathbf{1}_{\mathcal{O}_1} + \frac{\alpha_2}{N_2}\rho\mathbf{1}_{\mathcal{O}_2}\right|^2 \,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

$$\leq C(\tau_0,\tau_1,d_0)s^3\lambda^4\int_0^T\int_{\omega_1} \mathrm{e}^{-2s\eta}\varphi^3|\phi|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+ C(\tau_0,\tau_1,d_0,N_1,N_2,\alpha_1,\alpha_2)\int_Q s^2\lambda^4\varphi^3\mathrm{e}^{-2s\eta}|\rho|^2\,\mathrm{d}x\,\mathrm{d}t,$$

because $s,\ \lambda > 1$ and $\varphi^{-1}\in L^\infty(Q)$. Consequently,

$$\begin{aligned}\mathcal{K}(\rho)+\mathcal{K}(\phi) &\leq C(\tau_0,\tau_1,d_0)\left(s^3\lambda^4\int_0^T\int_{\omega_1}\mathrm{e}^{-2s\eta}\varphi^3|\rho|^2\,\mathrm{d}x\,\mathrm{d}t+\int_Q s^3\lambda^3\varphi^3\mathrm{e}^{-2s\eta}|\phi|^2\,\mathrm{d}x\,\mathrm{d}t\right)\\ &\quad + C(\tau_0,\tau_1,d_0)s^3\lambda^3\int_Q \mathrm{e}^{-2s\eta}\varphi^3|\phi|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + C(\tau_0,\tau_1,d_0,N_1,N_2,\alpha_1,\alpha_2)\int_Q s^2\lambda^4\varphi^3\mathrm{e}^{-2s\eta}|\rho|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We set $s_3=\max(s_2, 2C(\tau_0,\tau_1,d_0,N_1,N_2,\alpha_1,\alpha_2))$ and $\lambda_2=\max(\lambda_1,2C(\tau_0,\tau_1,d_0))$. Then if we choose in this latter inequality, $s\geq s_3$ and $\lambda\geq\lambda_2$, then (60) holds true.

Step 2. Now, we want to eliminate the local term corresponding to ϕ on the right-hand side of the estimate (60).

So, let ω_2 be a nonempty open set such that $\omega_1\subset\omega_2\subset\mathcal{O}_d\cap\omega$. Introduce as in [27] the cut off function $\xi\in C_0^\infty(\Omega)$ such that

$$0\leq\xi\leq 1,\quad \xi=1 \text{ in } \omega_1,\quad \xi=0 \text{ in } \Omega\setminus\omega_2, \tag{61a}$$
$$\frac{\Delta\xi}{\xi^{1/2}}\in L^\infty(\omega_2),\quad \frac{\nabla\xi}{\xi^{1/2}}\in[L^\infty(\omega_2)]^n. \tag{61b}$$

Set $u=s^3\lambda^4\varphi^3\mathrm{e}^{-2s\eta}$. Then $u(T)=u(0)=0$ and we have

$$\frac{\partial u}{\partial t}=u\left[3\varphi^{-1}\frac{\partial\varphi}{\partial t}-2s\frac{\partial\eta}{\partial t}\right], \tag{62a}$$
$$\nabla(u\xi)=u\left[(3\lambda+2s\lambda\varphi)\xi\nabla\psi+\nabla\xi\right] \tag{62b}$$

and

$$\begin{aligned}\Delta(u\xi)&=u\xi(14s\lambda^2\varphi+4s^2\lambda^2\varphi^2+9\lambda^2)|\nabla\psi|^2+u\xi\Delta\psi(3\lambda+2s\lambda\varphi)\\ &\quad+2u(3\lambda+2s\lambda\varphi)\nabla\psi.\nabla\xi+u\Delta\xi.\end{aligned} \tag{63}$$

If we multiply the first equation of (58) by $u\xi\phi$ and integrate by parts over Q, we obtain

$$\begin{aligned}&-\frac{\alpha_1}{N_1}\int_Q u\xi|\rho|^2\mathbf{1}_{\mathcal{O}_1}\,\mathrm{d}x\,\mathrm{d}t-\frac{\alpha_2}{N_2}\int_Q u\xi|\rho|^2\mathbf{1}_{\mathcal{O}_2}\,\mathrm{d}x\,\mathrm{d}t+\int_Q\rho\xi\phi\frac{\partial u}{\partial t}\,\mathrm{d}x\,\mathrm{d}t\\ &-2\int_Q\rho\nabla\phi.\nabla(u\xi)\,\mathrm{d}x\,\mathrm{d}t-\int_Q\rho\phi\Delta(u\xi)\,\mathrm{d}x\,\mathrm{d}t=\int_Q u\xi|\phi|^2\mathbf{1}_{\mathcal{O}_d}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

If we set

$$J_1 = -\frac{\alpha_1}{N_1}\int_Q u\xi|\rho|^2\mathbf{1}_{\mathcal{O}_1}\,\mathrm{d}x\,\mathrm{d}t - \frac{\alpha_2}{N_2}\int_Q u\xi|\rho|^2\mathbf{1}_{\mathcal{O}_2}\,\mathrm{d}x\,\mathrm{d}t, \quad J_2 = \int_Q \rho\xi\phi\frac{\partial u}{\partial t}\,\mathrm{d}x\,\mathrm{d}t,$$
$$J_3 = -2\int_Q \rho\nabla\phi.\nabla(u\xi)\,\mathrm{d}x\,\mathrm{d}t, \quad J_4 = -\int_Q \rho\phi\Delta(u\xi)\,\mathrm{d}x\,\mathrm{d}t,$$

the formula (64) can be rewritten as

$$J_1 + J_2 + J_3 + J_4 = \int_Q u|\phi|^2\mathbf{1}_{\mathcal{O}_d}\,\mathrm{d}x\,\mathrm{d}t. \tag{64}$$

Let us estimate $J_i,\ i = 1,\dots,4$. We have

$$\begin{aligned} J_1 &= -\frac{\alpha_1}{N_1}\int_Q u\xi|\rho|^2\mathbf{1}_{\mathcal{O}_1}\,\mathrm{d}x\,\mathrm{d}t - \frac{\alpha_2}{N_2}\int_Q u\xi|\rho|^2\mathbf{1}_{\mathcal{O}_2}\,\mathrm{d}x\,\mathrm{d}t \\ &\le \left(\frac{\alpha_1}{N_1} + \frac{\alpha_2}{N_2}\right)\int_Q s^3\lambda^4\varphi^3\mathrm{e}^{-2s\eta}|\rho|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Using the Young inequality, (48a), (61a), (62a) and (63), we obtain that

$$\begin{aligned} J_2 &= \int_Q \rho\xi\phi\frac{\partial u}{\partial t}\,\mathrm{d}x\,\mathrm{d}t \\ &\le \frac{\gamma_1}{2}\int_Q \xi u|\phi|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2\gamma_1}\int_Q \xi u|\rho|^2\left[18\varphi^{-2}\left(\frac{\partial\varphi}{\partial t}\right)^2 + 8s^2\left(\frac{\partial\eta}{\partial t}\right)^2\right]\mathrm{d}x\,\mathrm{d}t \\ &\le \frac{\gamma_1}{2}\int_0^T\int_{\omega_1} u|\phi|^2\,\mathrm{d}x\,\mathrm{d}t + C(T)\int_0^T\int_{\omega_2} s^5\lambda^4\varphi^7\mathrm{e}^{-2s\eta}|\rho|^2\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

for some $\gamma_1 > 0$.

$$\begin{aligned} J_3 &= -2\int_Q \rho\nabla\phi.\nabla(u\xi)\,\mathrm{d}x\,\mathrm{d}t \\ &= -2\int_Q \rho\xi u(3\lambda + 2s\lambda\varphi)\nabla\psi.\nabla\phi\,\mathrm{d}x\,\mathrm{d}t - 2\int_Q \rho u\nabla\phi.\nabla\xi\,\mathrm{d}x\,\mathrm{d}t \\ &\le \frac{1}{2}\int_Q s\lambda\varphi\mathrm{e}^{-2s\eta}|\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}t + C(\tau_1)\int_0^T\int_{\omega_2} s^7\lambda^9\varphi^7\mathrm{e}^{-2s\eta}|\rho|^2\,\mathrm{d}x\,\mathrm{d}t. \\ J_4 &= -\int_Q \rho\phi\Delta(u\xi)\,\mathrm{d}x\,\mathrm{d}t, \\ &= -\int_Q \rho u\xi\phi(14s\lambda^2\varphi + 4s^2\lambda^2\varphi^2 + 9\lambda^2)|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t - \int_Q \rho u\xi\phi\Delta\psi(3\lambda + 2s\lambda\varphi)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - 2\int_Q \rho u\phi(3\lambda + 2s\lambda\varphi)\nabla\psi.\nabla\xi\,\mathrm{d}x\,\mathrm{d}t - \int_Q \rho u\phi\Delta\xi\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

which after some calculations gives

$$J_4 \le \sum_{i=2}^{5} \frac{\gamma_i}{2} \int_0^T \int_{\omega_1} u|\phi|^2 \,\mathrm{d}x\,\mathrm{d}t + C(\tau_1) \int_0^T \int_{\omega_2} s^7\lambda^8\varphi^7 \mathrm{e}^{-2s\eta}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t,$$

for some $\gamma_i > 0,\ i = 2, \dots, 5$. Finally, choosing the γ_i such that $\sum_{i=1}^{5} \frac{\gamma_i}{2} = \frac{1}{2}$, it follows from (64) that

$$\begin{aligned}\int_0^T \int_{\omega_1} s^3\lambda^4\varphi^3 \mathrm{e}^{-2s\eta}|\phi|^2 \,\mathrm{d}x\,\mathrm{d}t \le{}& \int_Q s\lambda\varphi \mathrm{e}^{-2s\eta}|\nabla\phi|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ C(\tau_1, T) \int_0^T \int_{\omega_2} s^7\lambda^9\varphi^7 \mathrm{e}^{-2s\eta}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ \left(\frac{\alpha_1}{N_1} + \frac{\alpha_2}{N_2}\right) \int_Q s^3\lambda^4\varphi^3 \mathrm{e}^{-2s\eta}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{65}$$

Combining (60) with (65), we deduce that

$$\begin{aligned}\mathcal{K}(\rho) + \mathcal{K}(\phi) \le{}& C(\tau_0, \tau_1, d_0) \int_Q s\lambda\varphi \mathrm{e}^{-2s\eta}|\nabla\phi|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ C(\tau_0, \tau_1, d_0, T) \int_0^T \int_{\omega_2} s^7\lambda^9\varphi^7 \mathrm{e}^{-2s\eta}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ \left(\frac{\alpha_1}{N_1} + \frac{\alpha_2}{N_2}\right) C(\tau_0, \tau_1, d_0) \int_Q s^3\lambda^4\varphi^3 \mathrm{e}^{-2s\eta}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Taking in this latter inequality $\lambda \ge \lambda_2 = \max(\lambda_1, 2C(\tau_0, \tau_1, d_0))$, we get

$$\begin{aligned}\mathcal{K}(\rho) + \mathcal{K}(\phi) \le{}& C(\tau_0, \tau_1, d_0, T) \int_0^T \int_{\omega_2} s^7\lambda^9\varphi^7 \mathrm{e}^{-2s\eta}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ \left(\frac{\alpha_1}{N_1} + \frac{\alpha_2}{N_2}\right) C(\tau_0, \tau_1, d_0) \int_Q s^3\lambda^4\varphi^3 \mathrm{e}^{-2s\eta}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Taking $N_i,\ (i = 1, 2)$ large enough, we can absorb the last term of the latter inequality in the left-hand side. Using the fact that $\omega_2 \subset \omega$, we deduce (59). ■

Now, we are going to establish the observability inequality of Carleman in the sense that the weight functions do not vanish at $t = 0$. We define the functions $\tilde{\varphi}$ and $\tilde{\eta}$ as follows:

$$\tilde{\varphi}(t, x) = \begin{cases} \varphi\left(\frac{T}{2}, x\right) & \text{if } t \in \left[0, \frac{T}{2}\right], \\ \varphi(t, x) & \text{if } t \in \left[\frac{T}{2}, T\right] \end{cases} \tag{66}$$

and

$$\tilde{\eta}(t, x) = \begin{cases} \eta\left(\frac{T}{2}, x\right) & \text{if } t \in \left[0, \frac{T}{2}\right], \\ \eta(t, x) & \text{if } t \in \left[\frac{T}{2}, T\right]. \end{cases} \tag{67}$$

Then in view of the definition of φ and η given by (46) and (47), the functions $\tilde{\varphi}(., x)$ and $\tilde{\eta}(., x)$ are positive functions of class $\mathcal{C}^1$ on $[0, T[$. From now on, we fix $\lambda = \lambda_2$ and $s = s_3$.

We have the following result.

Proposition 2.6: *Under the assumptions of Proposition* 2.5, *there exist positive constants* $s_3 \geq 1$ *and* $\lambda_2 \geq 1$ *and two positive weight functions* $\theta = \theta(t)$ *and* $\varpi = \varpi(t,x)$ *such that for any solution* (ρ, Ψ_i) *of* (55)–(56), *we have*

$$\|\rho(0,\cdot)\|^2_{L^2(\Omega)} + \int_Q \frac{1}{\varpi^2}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t + \sum_{i=1}^{2} \int_Q \theta^2|\Psi_i|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C \int_{\omega^T} |\rho|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{68}$$

for some $C = C(\|a\|_\infty, \tau_0, \tau_1, d_0, T) > 0$.

Proof: We proceed in two steps.

Step 1. We prove that there exist a constant $C = C(\|a\|_\infty, \tau_0, \tau_1, d_0) > 0$ and a positive weight function ϖ such that

$$\|\rho(0,\cdot)\|^2_{L^2(\Omega)} + \int_Q \frac{1}{\varpi^2}|\rho|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C \int_{\omega^T} |\rho|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{69}$$

Let us introduce a function $\beta \in \mathcal{C}^1([0,T])$ such that

$$0 \leq \beta \leq 1, \quad \beta(t) = 1 \quad \text{for } t \in [0, T/2], \ \beta(t) = 0 \quad \text{for } t \in [3T/4, T], \quad |\beta'(t)| \leq C/T. \tag{70}$$

For any $(t,x) \in Q$, we set

$$\zeta(t,x) = \beta(t)\mathrm{e}^{-r(T-t)}\rho(t,x),$$

where $r > 0$. Then in view (58), the function ζ is a solution of

$$\begin{cases} -\dfrac{\partial \zeta}{\partial t} - \Delta\zeta + a\zeta + r\zeta = \beta \mathrm{e}^{-r(T-t)}\phi \mathbf{1}_{\mathcal{O}_d} - \beta' \mathrm{e}^{-r(T-t)}\rho & \text{in } Q, \\ \zeta = 0 & \text{on } \Sigma, \\ \zeta(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{71}$$

If we multiply the first equation in (71) by ζ and integrate by parts over Q, we get

$$\begin{aligned} &\frac{1}{2}\|\zeta(0,\cdot)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla\zeta\|^2_{L^2(Q)} + r\|\zeta\|^2_{L^2(Q)} \\ &\quad \leq \frac{1}{2}\left(2\|a\|_\infty + 2\right)\|\zeta\|^2_{L^2(Q)} \\ &\qquad + \frac{1}{2}\int_0^{3T/4}\int_\Omega |\phi|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_{T/2}^{3T/4}\int_\Omega |\rho|^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Hence, choosing in this latter inequality $r = \|a\|_\infty + \frac{3}{2}$ and using the definition of ζ, we obtain that

$$\begin{aligned} &\int_\Omega |\rho(0,x)|^2 \,\mathrm{d}x + \int_0^{T/2}\int_\Omega |\nabla\rho|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^{T/2}\int_\Omega |\rho|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq C\left(\|a\|_\infty, T\right)\left(\int_0^{3T/4}\int_\Omega |\phi|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{T/2}^{3T/4}\int_\Omega |\rho|^2 \,\mathrm{d}x\,\mathrm{d}t\right). \end{aligned}$$

Now, using the fact that the functions $\tilde{\varphi}$ and $\tilde{\eta}$ defined by (66) and (67), respectively, have lower bounds for $(t,x) \in [0, T/2] \times \Omega$, we get

$$\int_\Omega |\rho(0,x)|^2 \,\mathrm{d}x + \widetilde{\mathcal{K}}_{[0,T/2]}(\rho)$$

$$\leq C\left(\|a\|_{\infty}, T\right)\left(\int_{0}^{3T/4}\int_{\Omega}|\phi|^{2}\,\mathrm{d}x\,\mathrm{d}t+\int_{T/2}^{3T/4}\int_{\Omega}|\rho|^{2}\,\mathrm{d}x\,\mathrm{d}t\right), \tag{72}$$

and

$$\widetilde{\mathcal{K}}_{[a,b]}(z)=\int_{a}^{b}\int_{\Omega}\mathrm{e}^{-2s_{3}\tilde{\eta}}\tilde{\varphi}|\nabla z|^{2}\,\mathrm{d}x\,\mathrm{d}t+\int_{a}^{b}\int_{\Omega}\mathrm{e}^{-2s_{3}\tilde{\eta}}\tilde{\varphi}^{3}|z|^{2}\,\mathrm{d}x\,\mathrm{d}t. \tag{73}$$

Adding the term $\widetilde{\mathcal{K}}_{[0,T/2]}(\phi)$ to both sides of inequality (72), we obtain

$$\begin{aligned}&\int_{\Omega}|\rho(0,x)|^{2}\,\mathrm{d}x+\widetilde{\mathcal{K}}_{[0,T/2]}(\rho)+\widetilde{\mathcal{K}}_{[0,T/2]}(\phi)\\&\quad\leq C\left(\|a\|_{\infty}, T\right)\left(\int_{0}^{3T/4}\int_{\Omega}|\phi|^{2}\,\mathrm{d}x\,\mathrm{d}t+\int_{T/2}^{3T/4}\int_{\Omega}|\rho|^{2}\,\mathrm{d}x\;\mathrm{d}t\right)+\widetilde{\mathcal{K}}_{[0,T/2]}(\phi).\end{aligned} \tag{74}$$

In order to eliminate the term $\widetilde{\mathcal{K}}_{[0,T/2]}(\phi)$ on the right-hand side of (74), we use the standard energy estimates of system (56) and we obtain

$$\int_{0}^{T/2}\int_{\Omega}|\nabla\phi|^{2}\,\mathrm{d}x\,\mathrm{d}t+\int_{0}^{T/2}\int_{\Omega}|\phi|^{2}\,\mathrm{d}x\,\mathrm{d}t\leq C\left(\frac{\alpha_{1}^{2}}{2N_{1}^{2}}+\frac{\alpha_{2}^{2}}{2N_{2}^{2}}\right)\int_{0}^{T/2}\int_{\Omega}|\rho|^{2}\,\mathrm{d}x\,\mathrm{d}t,$$

where $C=C(\|a\|_{\infty},T)>0$ is independent of $N_i,\ (i=1,2)$. Since the functions $\tilde{\varphi}$ and $\tilde{\eta}$ have lower and upper bounds for $(t,x)\in[0,T/2]\times\Omega$, from the previous inequality we obtain

$$\widetilde{\mathcal{K}}_{[0,T/2]}(\phi)\leq C\left(\|a\|_{\infty}, T\right)\left(\frac{\alpha_{1}^{2}}{2N_{1}^{2}}+\frac{\alpha_{2}^{2}}{2N_{2}^{2}}\right)\int_{0}^{T/2}\int_{\Omega}\mathrm{e}^{-2s_{3}\tilde{\eta}}\tilde{\varphi}^{3}\,|\rho|^{2}\,\mathrm{d}x\,\mathrm{d}t. \tag{75}$$

Replacing (75) in (74) and taking $N_i,\ (i=1,2)$ large enough, we obtain

$$\int_{\Omega}|\rho(0,x)|^{2}\,\mathrm{d}x+\widetilde{\mathcal{K}}_{[0,T/2]}(\rho)+\widetilde{\mathcal{K}}_{[0,T/2]}(\phi)\leq C\left(\|a\|_{\infty}, T\right)\left(\int_{T/2}^{3T/4}\int_{\Omega}(|\rho|^{2}+|\phi|^{2})\,\mathrm{d}x\,\mathrm{d}t\right). \tag{76}$$

Since the functions φ and η defined by (46) and (47), respectively, have the upper bound for $(t,x)\in[T/2,3T/4]\times\Omega$, using (59), we obtain

$$\begin{aligned}&\int_{\Omega}|\rho(0,x)|^{2}\,\mathrm{d}x+\widetilde{\mathcal{K}}_{[0,T/2]}(\rho)+\widetilde{\mathcal{K}}_{[0,T/2]}(\phi)\leq C\left(\|a\|_{\infty}, T\right)\left(\mathcal{K}(\rho)+\mathcal{K}(\phi)\right)\\&\qquad\leq C(\|a\|_{\infty},T,\tau_{0},\tau_{1},d_{0})\int_{\omega^{T}}\mathrm{e}^{-2s_{3}\eta}\varphi^{7}|\rho|^{2}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{77}$$

On the other hand, since $\eta=\tilde{\eta}$ and $\varphi=\tilde{\varphi}$ in $[T/2,T]\times\Omega$, using again estimate (59), we obtain

$$\begin{aligned}&\widetilde{\mathcal{K}}_{[T/2,T]}(\rho)+\widetilde{\mathcal{K}}_{[T/2,T]}(\phi)=\mathcal{K}(\rho)+\mathcal{K}(\phi)\\&\qquad\leq C(\|a\|_{\infty},T,\tau_{0},\tau_{1},d_{0})\int_{\omega^{T}}\mathrm{e}^{-2s_{3}\eta}\varphi^{7}|\rho|^{2}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{78}$$

Adding (77) and (78) and using the fact that $\mathrm{e}^{-2s_3\eta}\varphi^7\in L^{\infty}(Q)$, we deduce that

$$\|\rho(0,\cdot)\|_{L^{2}(\Omega)}^{2}+\widetilde{\mathcal{K}}_{[0,T]}(\rho)+\widetilde{\mathcal{K}}_{[0,T]}(\phi)\leq C(\|a\|_{\infty},T,\tau_{0},\tau_{1},d_{0})\int_{\omega^{T}}|\rho|^{2}\,\mathrm{d}x\,\mathrm{d}t. \tag{79}$$

Using the definition of $\widetilde{\mathcal{K}}_{[a,b]}$ given by (73), we can rewrite the inequality (79) as

$$\|\rho(0,\cdot)\|_{L^{2}(\Omega)}^{2}+\int_{0}^{T}\int_{\Omega}\mathrm{e}^{-2s_{3}\tilde{\eta}}\tilde{\varphi}^{3}|\rho|^{2}\,\mathrm{d}x\,\mathrm{d}t+\int_{0}^{T}\int_{\Omega}\mathrm{e}^{-2s_{3}\tilde{\eta}}\tilde{\varphi}^{3}|\phi|^{2}\,\mathrm{d}x\,\mathrm{d}t\leq C\int_{\omega^{T}}|\rho|^{2}\,\mathrm{d}x\,\mathrm{d}t, \tag{80}$$

where $C=C(\|a\|_{\infty},T,\tau_{0},\tau_{1},d_{0})>0$.

We set

$$\frac{1}{\varpi^2} = \mathrm{e}^{-2s_3\tilde{\eta}}\tilde{\varphi}^3. \tag{81}$$

Then, from the definition of $\tilde{\varphi}$ and $\tilde{\eta}$ given, respectively, by (66) and (67), we have that $\frac{1}{\varpi^2} \in L^\infty(Q)$. Using (80) and (81), we deduce (69).

Step 2. We prove that there exist a constant $C = C(\|a\|_\infty, \tau_0, \tau_1, d_0) > 0$ and a positive weight function θ such that

$$\sum_{i=1}^{2} \int_Q \theta^2 |\Psi_i|^2 \, \mathrm{d}x \, \mathrm{d}t \leq C \int_{\omega^T} |\rho|^2 \, \mathrm{d}x \, \mathrm{d}t. \tag{82}$$

We set

$$\eta_0(t) = \max_{x\in\Omega} \tilde{\eta}(t,x). \tag{83}$$

Then, η_0 is also a positive function of class $\mathcal{C}^1$ on $[0, T[$. We define the weight function θ by:

$$\theta(t) = \mathrm{e}^{-s_3\eta_0(t)} \in L^\infty(0,T). \tag{84}$$

Multiplying the first equation of (56) by $\theta^2\Psi_i$, $i = 1, 2$ and integrating by parts over Ω, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta^2|\Psi_i|^2 \, \mathrm{d}x + \int_\Omega \theta^2 |\nabla \Psi_i|^2 \, \mathrm{d}x \\
&\quad = -\int_\Omega \theta^2 \, a \, |\Psi_i|^2 \, \mathrm{d}x - \frac{1}{N_i}\int_{\mathcal{O}_i} \theta^2 \rho \Psi_i \, \mathrm{d}x - s_3 \int_\Omega \theta^2 \frac{\partial \eta_0}{\partial t} |\Psi_i|^2 \, \mathrm{d}x.
\end{aligned}$$

Hence, using the fact that $\partial\eta_0/\partial t$ is a positive function on $[0, T)$, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta^2|\Psi_i|^2 \, \mathrm{d}x + \frac{1}{2}\int_\Omega \theta^2|\nabla\Psi_i|^2 \, \mathrm{d}x \leq \left(\|a\|_\infty + \frac{1}{2}\right)\int_\Omega \theta^2|\Psi_i|^2 \, \mathrm{d}x + \frac{1}{2N_i^2}\int_{\mathcal{O}_i} \theta^2|\rho|^2 \, \mathrm{d}x.$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\theta\Psi_i|^2 \, \mathrm{d}x\right) \leq (2\|a\|_\infty + 1)\int_\Omega |\theta\Psi_i|^2 \, \mathrm{d}x + \frac{1}{N_i^2}\int_{\mathcal{O}_i} \theta^2|\rho|^2 \, \mathrm{d}x.$$

Using Gronwall's Lemma and the fact that $\Psi_i(x, 0) = 0$ for $x \in \Omega$, we obtain that

$$\int_\Omega \theta^2|\Psi_i(x,t)|^2 \, \mathrm{d}x \leq C_i \int_Q \theta^2|\rho|^2 \, \mathrm{d}x \, \mathrm{d}t, \quad \text{for all } t \in [0,T], \tag{85}$$

where $C_i = C(\|a\|_\infty, T, N_i) = \mathrm{e}^{(2\|a\|_\infty+1)T}\frac{1}{N_i^2} > 0, \quad i = 1, 2.$

In view of (83), (84) and the fact that $\tilde{\varphi}^{-1} \in L^\infty(Q)$, we have that

$$\int_Q \theta^2|\rho(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \int_Q \mathrm{e}^{-2s_3\tilde{\eta}}\tilde{\varphi}^3|\rho(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t,$$

which combining with (85) and (69) yields

$$\int_Q \theta^2|\Psi_i(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \leq C\int_{\omega^T} |\rho(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t, \quad i = 1, 2,$$

where $C = C(\|a\|_\infty, T, N_1, N_2, \tau_0, \tau_1, d_0) > 0$. Hence we deduce (82).

Finally, adding estimates (69) and (82), we obtain (68). ■

2.3. Null controllability (Problem 1.2 for system (16))

In this subsection, we will achieve the proof of Theorem 1.1. We look for a control $h \in L^2(\omega^T)$ such that the solution $(\hat{y}, \hat{p}_i)$ of

$$\begin{cases} \dfrac{\partial \hat{y}}{\partial t} - \Delta \hat{y} + a\hat{y} = h\mathbf{1}_\omega - \dfrac{1}{N_1}\hat{p}_1\mathbf{1}_{\mathcal{O}_1} - \dfrac{1}{N_2}\hat{p}_2\mathbf{1}_{\mathcal{O}_2} & \text{in } Q, \\ \hat{y} = 0 & \text{on } \Sigma, \\ \hat{y}(0,.) = y^0 & \text{in } \Omega \end{cases} \tag{86}$$

and

$$\begin{cases} -\dfrac{\partial \hat{p}_i}{\partial t} - \Delta \hat{p}_i + a\hat{p}_i = \alpha_i(\hat{y} - z_{i,d})\mathbf{1}_{\mathcal{O}_{i,d}} & \text{in } Q, \\ \hat{p}_i = 0 & \text{on } \Sigma, \\ \hat{p}_i(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{87}$$

satisfies (7).

To prove this null controllability problem, we proceed in three steps using a penalization method.

Step 1. For any $\varepsilon > 0$, we define the cost function:

$$J_\varepsilon(h) = \frac{1}{2\varepsilon}\|\hat{y}(T,.;h)\|_{L^2(\Omega)} + \frac{1}{2}\|h\|_{L^2(\omega^T)}. \tag{88}$$

Then we consider the optimal control problem:

$$\min_{h\in L^2(\omega^T)} J_\varepsilon(h). \tag{89}$$

Using minimizing sequences, we prove that there exists a unique solution $\hat{h}_\varepsilon \in L^2(\omega^T)$ to (89). Using Euler–Lagrange first-order optimality condition that characterizes the solution $\hat{h}_\varepsilon$, we can prove that

$$\hat{h}_\varepsilon = \hat{\rho}_\varepsilon \quad \text{in } \omega_T. \tag{90}$$

where $\hat{\rho}_\varepsilon$ is solution of

$$\begin{cases} -\dfrac{\partial \hat{\rho}_\varepsilon}{\partial t} - \Delta \hat{\rho}_\varepsilon + a\hat{\rho}_\varepsilon = \displaystyle\sum_{i=1}^{2} \alpha_i \hat{\Psi}_{i\varepsilon}\mathbf{1}_{\mathcal{O}_d} & \text{in } Q, \\ \hat{\rho}_\varepsilon = 0 & \text{on } \Sigma, \\ \hat{\rho}_\varepsilon(T,.) = -\dfrac{1}{\varepsilon}\hat{y}_\varepsilon(T,.) & \text{in } \Omega, \end{cases} \tag{91}$$

with $(\hat{\Psi}_{i\varepsilon}, \hat{y}_\varepsilon, \hat{p}_{i\varepsilon})$ are solutions of

$$\begin{cases} \dfrac{\partial \hat{\Psi}_{i\varepsilon}}{\partial t} - \Delta \hat{\Psi}_{i\varepsilon} + a\hat{\Psi}_{i\varepsilon} = -\dfrac{1}{N_i}\hat{\rho}_\varepsilon\mathbf{1}_{\mathcal{O}_i} & \text{in } Q, \\ \hat{\Psi}_\varepsilon = 0 & \text{on } \Sigma, \\ \hat{\Psi}_\varepsilon(0,.) = 0 & \text{in } \Omega, \end{cases} \tag{92}$$

$$\begin{cases} \dfrac{\partial \hat{y}_\varepsilon}{\partial t} - \Delta \hat{y}_\varepsilon + a\hat{y}_\varepsilon = h_\varepsilon\mathbf{1}_\omega - \dfrac{1}{N_1}\hat{p}_{1\varepsilon}\mathbf{1}_{\mathcal{O}_1} - \dfrac{1}{N_2}\hat{p}_{2\varepsilon}\mathbf{1}_{\mathcal{O}_2} & \text{in } Q, \\ \hat{y}_\varepsilon = 0 & \text{on } \Sigma, \\ \hat{y}_\varepsilon(0,.) = y^0 & \text{in } \Omega \end{cases} \tag{93}$$

and

$$\begin{cases} -\dfrac{\partial \hat{p}_{i\varepsilon}}{\partial t} - \Delta \hat{p}_{i\varepsilon} + a\hat{p}_{i\varepsilon} = \alpha_i(\hat{y}_\varepsilon - z_{i,d})\mathbf{1}_{\mathcal{O}_d} & \text{in } Q, \\ \hat{p}_{i\varepsilon} = 0 & \text{on } \Sigma, \\ \hat{p}_{i\varepsilon}(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{94}$$

Step 2. We give estimates on $\hat{y}_\varepsilon, \hat{p}_{1\varepsilon}, \hat{p}_{2\varepsilon}, \hat{h}_\varepsilon$ independent on ε.

If we multiply the first equation of (93) by ρ_ε solution of (91) and the first equation of (94) by $\Psi_{i\varepsilon}$, $i = 1, 2$ solution of (92) and integrate by parts over Q and use (90), we obtain that

$$\begin{aligned} &-\sum_{i=1}^{2} \frac{1}{N_i} \int_0^T \int_{\mathcal{O}_i} \hat{p}_{i\varepsilon} \hat{\rho}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \|\hat{h}_\varepsilon\|^2_{L^2(\omega^T)} \\ &\quad = -\frac{1}{\varepsilon}\|\hat{y}_\varepsilon(T,\cdot)\|^2_{L^2(\Omega)} - \int_\Omega y^0 \hat{\rho}_\varepsilon(0,x)\,\mathrm{d}x + \sum_{i=1}^{2} \alpha_i \int_{\mathcal{O}_d^T} \hat{y}_\varepsilon \hat{\Psi}_{i\varepsilon}\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{95}$$

and

$$-\frac{1}{N_1}\int_0^T \int_{\mathcal{O}_1} \hat{p}_{1\varepsilon}\hat{\rho}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \alpha_1 \int_{\mathcal{O}_d^T} \hat{y}_\varepsilon \hat{\Psi}_{1\varepsilon}\,\mathrm{d}x\,\mathrm{d}t - \alpha_1 \int_{\mathcal{O}_d^T} z_{1,d}\hat{\Psi}_{1\varepsilon}\,\mathrm{d}x\,\mathrm{d}t, \tag{96a}$$

$$-\frac{1}{N_2}\int_0^T \int_{\mathcal{O}_2} \hat{p}_{2\varepsilon}\hat{\rho}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \alpha_2 \int_{\mathcal{O}_d^T} \hat{y}_\varepsilon \hat{\Psi}_{2\varepsilon}\,\mathrm{d}x\,\mathrm{d}t - \alpha_2 \int_{\mathcal{O}_d^T} z_{2,d}\hat{\Psi}_{2\varepsilon}\,\mathrm{d}x\,\mathrm{d}t. \tag{96b}$$

Adding (96a) to (96b), then combining the result with (95), we deduce that

$$\|\hat{h}_\varepsilon\|^2_{L^2(\omega^T)} + \frac{1}{\varepsilon}\|\hat{y}_\varepsilon(T,\cdot)\|^2_{L^2(\Omega)} = \sum_{i=1}^{2}\alpha_i \int_{\mathcal{O}_d^T} z_{i,d}\hat{\Psi}_{i\varepsilon}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega y^0 \hat{\rho}_\varepsilon(0,x)\,\mathrm{d}x,$$

which using the Cauchy–Schwarz inequality and the fact that $\frac{1}{\theta}z_{i,d} \in L^2(\mathcal{O}_d^T)$ gives

$$\begin{aligned} \|\hat{h}_\varepsilon\|^2_{L^2(\omega^T)} + \frac{1}{\varepsilon}\|\hat{y}_\varepsilon(T,\cdot)\|^2_{L^2(\Omega)} &\le \sum_{i=1}^{2}\alpha_i \left\|\frac{1}{\theta}z_{i,d}\right\|_{L^2(\mathcal{O}_d^T)} \left\|\theta\hat{\Psi}_{i\varepsilon}\right\|_{L^2(Q)} \\ &\quad + \|y^0\|_{L^2(\Omega)}\|\hat{\rho}_\varepsilon(0,.)\|_{L^2(\Omega)}. \end{aligned}$$

This implies that

$$\begin{aligned} \|\hat{h}_\varepsilon\|^2_{L^2(\omega^T)} + \frac{1}{\varepsilon}\|\hat{y}_\varepsilon(T,\cdot)\|^2_{L^2(\Omega)} &\le \left(\sum_{i=1}^{2}\alpha_i^2 \left\|\frac{1}{\theta}z_{i,d}\right\|^2_{L^2(\mathcal{O}_d^T)} + \|y^0\|^2_{L^2(\Omega)}\right)^{1/2} \\ &\quad \times \left(\sum_{i=1}^{2}\left\|\theta\hat{\Psi}_{i\varepsilon}\right\|^2_{L^2(Q)} + \|\hat{\rho}_\varepsilon(0,.)\|^2_{L^2(\Omega)}\right)^{1/2}. \end{aligned} \tag{97}$$

From which we deduce that

$$\|\hat{h}_\varepsilon\|^2_{L^2(\omega^T)} \le \left(\sum_{i=1}^{2}\alpha_i^2 \left\|\frac{1}{\theta}z_{i,d}\right\|^2_{L^2(\mathcal{O}_d^T)} + \|y^0\|^2_{L^2(\Omega)}\right)^{1/2}$$

$$\times\left(\sum_{i=1}^{2}\left\|\theta\hat{\Psi}_{i\varepsilon}\right\|^2_{L^2(Q)}+\|\hat{\rho}_\varepsilon(0,.)\|^2_{L^2(\Omega)}\right)^{1/2}. \tag{98}$$

Using the observability inequality (68) to ρ_ε and $\Psi_{i\varepsilon}$, $i=1,2$ solutions of (91) and (92), we get

$$\sum_{i=1}^{2}\int_Q \theta^2|\hat{\Psi}_{i\varepsilon}|^2\,\mathrm{d}x\,\mathrm{d}t+\|\hat{\rho}_\varepsilon(0,.)\|^2_{L^2(\Omega)}\le C\int_{\omega^T}|\hat{\rho}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{99}$$

for some $C=C(\|a\|_\infty,\tau_0,\tau_1,d_0,T)>0$. Therefore combining (98) with (99), then using the fact that (90) holds true, we have that there exists a constant $C=C(\|a\|_\infty,\tau_0,\tau_1,d_0,T)>0$ such that

$$\|\hat{h}_\varepsilon\|^2_{L^2(\omega^T)}\le C\left(\sum_{i=1}^{2}\alpha_i^2\left\|\frac{1}{\theta}z_{i,d}\right\|^2_{L^2(\mathcal{O}_d^T)}+\|y^0\|^2_{L^2(\Omega)}\right)^{1/2}\|h_\varepsilon\|_{L^2(\omega^T)}.$$

Hence, we deduce that there exists $C=C(\|a\|_\infty,\tau_0,\tau_1,d_0,T)>0$ such that

$$\|\hat{h}_\varepsilon\|_{L^2(\omega^T)}\le C\left(\sum_{i=1}^{2}\left\|\frac{1}{\theta}z_{i,d}\right\|^2_{L^2(\mathcal{O}_d^T)}+\|y^0\|^2_{L^2(\Omega)}\right)^{1/2}, \tag{100}$$

and it follows from (90) that

$$\|\hat{\rho}_\varepsilon\|_{L^2(\omega^T)}\le C(\|a\|_\infty,\tau_0,\tau_1,d_0,T)\left(\sum_{i=1}^{2}\left\|\frac{1}{\theta}z_{i,d}\right\|^2_{L^2(\mathcal{O}_d^T)}+\|y^0\|^2_{L^2(\Omega)}\right)^{1/2}. \tag{101}$$

Using again (97) we have

$$\begin{aligned}\frac{1}{\varepsilon}\|\hat{y}_\varepsilon(T,\cdot)\|^2_{L^2(\Omega)}\le&\left(\sum_{i=1}^{2}\alpha_i^2\left\|\frac{1}{\theta}z_{i,d}\right\|^2_{L^2(\mathcal{O}_d^T)}+\|y^0\|^2_{L^2(\Omega)}\right)^{1/2}\\&\times\left(\sum_{i=1}^{2}\left\|\theta\hat{\Psi}_{i\varepsilon}\right\|^2_{L^2(Q)}+\|\hat{\rho}_\varepsilon(0,.)\|^2_{L^2(\Omega)}\right)^{1/2},\end{aligned}$$

which combining with (99) and (101) gives

$$\|\hat{y}_\varepsilon(T,\cdot)\|_{L^2(\Omega)}\le C(\|a\|_\infty,\tau_0,\tau_1,d_0,T)\sqrt{\varepsilon}\left(\sum_{i=1}^{2}\left\|\frac{1}{\theta}z_{i,d}\right\|^2_{L^2(\mathcal{O}_d^T)}+\|y^0\|^2_{L^2(\Omega)}\right)^{1/2}. \tag{102}$$

In view of (27), (40) and (100), we have that

$$\begin{aligned}\|\hat{v}_{i\varepsilon}\|_{L^2(\mathcal{O}_i^T)}&=\left\|-\frac{1}{N_i}\hat{p}_{i\varepsilon}\right\|_{L^2(\mathcal{O}_i^T)}\\&\le C(\|a\|_\infty,T,\alpha_1,\alpha_2,N_1,N_2)\left(\sum_{i=1}^{2}\|z_{i,d}\|_{L^2(\mathcal{O}_d^T)}+\|\hat{h}_\varepsilon\|_{L^2(\omega^T)}+\|y^0\|_{L^2(\Omega)}\right)\\&\le C\left(\sum_{i=1}^{2}\|z_{i,d}\|_{L^2(\mathcal{O}_d^T)}+\sum_{i=1}^{2}\left\|\frac{1}{\theta}z_{i,d}\right\|_{L^2(\mathcal{O}_d^T)}+\|y^0\|_{L^2(\Omega)}\right),\end{aligned}$$

where $C = C(\|a\|_\infty, \alpha_1, \alpha_2, N_1, N_2, \tau_0, \tau_1, d_0, T) > 0$. Hence, for $i = 1, 2$, we have that

$$\left\| -\frac{1}{N_i}\hat{p}_{i\varepsilon} \right\|_{L^2(\mathcal{O}_i^T)} \leq C\left(\sum_{i=1}^{2} \|z_{i,d}\|_{L^2(\mathcal{O}_d^T)} + \sum_{i=1}^{2} \left\| \frac{1}{\theta} z_{i,d} \right\|_{L^2(\mathcal{O}_d^T)} + \|y^0\|_{L^2(\Omega)} \right), \tag{103}$$

where $C = C(\|a\|_\infty, \alpha_1, \alpha_2, N_1, N_2, \tau_0, \tau_1, d_0, T) > 0$. Using (103), (100), (101), (93), (94), (92), we prove that there exists $C = C(\|a\|_\infty, \alpha_1, \alpha_2, N_1, N_2, \tau_0, \tau_1, d_0, T) > 0$ such that

$$\|\hat{y}_\varepsilon\|_{L^2((0,T);H_0^1(\Omega))} \leq C\left(\|y^0\|_{L^2(\Omega)} + \sum_{i=1}^{2} \|z_{i,d}\|_{L^2(\mathcal{O}_d^T)} + \sum_{i=1}^{2} \left\| \frac{1}{\theta} z_{i,d} \right\|_{L^2(\mathcal{O}_d^T)} \right), \tag{104a}$$

$$\|\hat{p}_{i\varepsilon}\|_{L^2((0,T);H_0^1(\Omega))} \leq C\left(\|y^0\|_{L^2(\Omega)} + \sum_{i=1}^{2} \|z_{i,d}\|_{L^2(\mathcal{O}_d^T)} + \sum_{i=1}^{2} \left\| \frac{1}{\theta} z_{i,d} \right\|_{L^2(\mathcal{O}_d^T)} \right). \tag{104b}$$

Note that (104b) is valid for $i = 1, 2$.

Step3. We study the convergence when $\varepsilon \to 0$ to the sequences $\hat{h}_\varepsilon$, $\hat{y}_\varepsilon$, $\hat{p}_{i\varepsilon}$, $i = 1, 2$ $\hat{\Psi}_{i\varepsilon}$, $i = 1, 2$ and $\hat{\rho}_\varepsilon$.

In view of (100), (103), (104a) and (102), we can extract subsequences still denoted by $\hat{h}_\varepsilon$, $\hat{y}_\varepsilon$ and $\hat{p}_{i\varepsilon}$ such that when $\varepsilon \to 0$, we have

$$\hat{h}_\varepsilon \rightharpoonup \hat{h} \text{ weakly in } \quad L^2(\omega^T), \tag{105a}$$

$$\hat{y}_\varepsilon \rightharpoonup \hat{y} \text{ weakly in } \quad L^2((0,T); H_0^1(\Omega)), \tag{105b}$$

$$\hat{p}_{i\varepsilon} \rightharpoonup \hat{p}_i \text{ weakly in } \quad L^2((0,T); H_0^1(\Omega)), \tag{105c}$$

$$\hat{y}_\varepsilon(T, .) \to 0 \text{ strongly in } \quad L^2(\Omega). \tag{105d}$$

From (103) and (105c), we obtain that

$$-\frac{1}{N_i}\hat{p}_{i\varepsilon} \rightharpoonup \hat{v}_i = -\frac{\hat{p}_i}{N_i} \quad \text{in } \mathcal{O}_i^T, \quad i = 1, 2. \tag{106}$$

Moreover, using the weak lower semi-continuity of the norm, we deduce from (105c), (106) and (103) that

$$\|\hat{v}_i\|_{L^2(\mathcal{O}_i^T)} \leq C\left(\|y^0\|_{L^2(\Omega)} + \sum_{i=1}^{2} \|z_{i,d}\|_{L^2(\mathcal{O}_d^T)} + \sum_{i=1}^{2} \left\| \frac{1}{\theta} z_{i,d} \right\|_{L^2(\mathcal{O}_d^T)} \right), \tag{107}$$

where $C = C(\|a\|_\infty, \alpha_1, \alpha_2, N_1, N_2, \tau_0, \tau_1, d_0, T) > 0$.

Let $\mathcal{D}(Q)$ be the set of infinitely continuously differentiable functions with compact support on Q. If we multiply the first equation in (93) by $\Phi \in \mathcal{D}(Q)$ and the first equation in (94) by $\xi_i \in \mathcal{D}(Q)$, $i = 1, 2$, and integrate by parts over Q and then take the limit when $\varepsilon \to 0$ while using (105a), we, respectively, deduce that

$$\int_Q \hat{y}\left(-\frac{\partial \Phi}{\partial t} - \Delta\Phi + a\Phi \right) \mathrm{d}x\,\mathrm{d}t = -\sum_{i=1}^{2} \frac{1}{N_i} \int_{\mathcal{O}_i^T} \hat{p}_i \Phi \ \mathrm{d}x\,\mathrm{d}t + \int_{\omega^T} \hat{h}\Phi \ \mathrm{d}x\,\mathrm{d}t,$$

and

$$\int_Q \hat{p}_i \left(\frac{\partial \xi_i}{\partial t} - \Delta\xi_i + a\xi_i \right) \mathrm{d}x\,\mathrm{d}t = \alpha_i \int_Q (\hat{y} - z_{i,d}) \mathbf{1}_{\mathcal{O}_d} \xi_i \ \mathrm{d}x\,\mathrm{d}t, \quad i = 1, 2,$$

which after an integration by parts over Q, gives, respectively

$$\int_Q \left(\frac{\partial \hat{y}}{\partial t} - \Delta \hat{y} + a\hat{y}\right) \Phi \, \mathrm{d}x\,\mathrm{d}t = -\sum_{i=1}^{2} \frac{1}{N_i} \int_{\mathcal{O}_i^T} \hat{p}_i \Phi \, \mathrm{d}x\,\mathrm{d}t + \int_{\omega^T} \hat{h}\Phi \, \mathrm{d}x\,\mathrm{d}t, \quad \text{for every } \Phi \in \mathcal{D}(Q),$$

and

$$\int_Q \left(-\frac{\partial \hat{p}_i}{\partial t} - \Delta \hat{p}_i + a\hat{p}_i\right) \xi_i \, \mathrm{d}x\,\mathrm{d}t = \alpha_i \int_Q (\hat{y} - z_{i,d})\mathbf{1}_{\mathcal{O}_d} \xi_i \, \mathrm{d}x\,\mathrm{d}t, \quad \text{for every } \xi_i \in \mathcal{D}(Q), \quad i = 1, 2.$$

Hence, we deduce that

$$\frac{\partial \hat{y}}{\partial t} - \Delta \hat{y} + a\hat{y} = \sum_{i=1}^{2} \frac{1}{N_i} \hat{p}_i \mathbf{1}_{\mathcal{O}_i} + \hat{h}\mathbf{1}_\omega \quad \text{in } Q, \tag{108a}$$

$$-\frac{\partial \hat{p}_i}{\partial t} - \Delta \hat{p}_i + a\hat{p}_i = \alpha_i(\hat{y} - z_{i,d})\mathbf{1}_{\mathcal{O}_d} \quad \text{in } Q, \quad i = 1, 2. \tag{108b}$$

Observing that $\hat{y}$, $\hat{p}_1, \hat{p}_2 \in L^2((0,T); H_0^1(\Omega))$ and $\frac{\partial \hat{y}}{\partial t}$, $\frac{\partial \hat{p}_1}{\partial t}$ and $\frac{\partial \hat{p}_2}{\partial t}$ belong to $L^2((0,T); H^{-1}(\Omega))$ we deduce that $\hat{y}(0)$, $\hat{y}(T)$, $\hat{p}_1(T)$ and $\hat{p}_2(T)$ exists in $L^2(\Omega)$. The traces of $\hat{y}(t)$, $\hat{p}_1(t)$ and $\hat{p}_2(t)$ exist in $L^2(\Gamma)$ for almost every $t \in (0, T)$. Therefore passing to the limit in the second and third equations of (93) and (94), we obtained from (105b) and (105c) that

$$\hat{y} = 0 \quad \text{on } \Sigma, \tag{109a}$$

$$\hat{p}_i = 0 \quad \text{on } \Sigma, \quad i = 1, 2, \tag{109b}$$

$$\hat{p}_i(T, .) = 0 \quad \text{in } \Omega, \quad i = 1, 2, \tag{109c}$$

$$\hat{y}(0, .) = y^0 \quad \text{in } \Omega; \tag{109d}$$

and it follows from (102) that

$$\tilde{y}(T, .) = 0 \text{ in } \Omega. \tag{110}$$

Thus $\hat{y} = \hat{y}(t, x; \hat{h}, \hat{v}_1, \hat{v}_2))$ and $\hat{p}_i = \hat{p}_i(\hat{h})$, $i = 1, 2$ are solutions of (86) and (87).

If we apply the Carleman inequality (68) to $\hat{\rho}_\varepsilon$ and $\hat{\Psi}_{i\varepsilon}$, $i = 1, 2$, we deduce that there exists $C = C(\|a\|_\infty, \tau_0, \tau_1, d_0, T) > 0$ such that

$$\int_Q \frac{1}{\varpi^2} |\hat{\rho}_\varepsilon|^2 \, \mathrm{d}x\,\mathrm{d}t + \sum_{i=1}^{2} \int_Q \theta^2 |\hat{\Psi}_{i,\varepsilon}|^2 \, \mathrm{d}x\,\mathrm{d}t \le \int_{\omega^T} |\hat{\rho}_\varepsilon|^2 \, \mathrm{d}x\,\mathrm{d}t$$
$$\le C\left(\sum_{i=1}^{2} \left\| \frac{1}{\theta} z_{i,d} \right\|^2_{L^2(\mathcal{O}_d^T)} + \|y^0\|^2_{L^2(\Omega)}\right),$$

because (101) holds true. Hence in view of the definitions of ϖ and θ given by (81) and (84), it can be readily seen that there exists a constant $C > 0$ such that

$$\theta^2 \ge C \text{ in } Q \quad \text{and} \quad \frac{1}{\varpi^2} \ge C \text{ in } Q \tag{111}$$

and therefore we can obtain

$$\left\| \hat{\Psi}_{i\varepsilon} \right\|^2_{L^2(Q)} + \left\| \hat{\rho}_\varepsilon \right\|^2_{L^2(Q)} \le C\left(\sum_{i=1}^{2} \left\| \frac{1}{\theta} z_{i,d} \right\|^2_{L^2(\mathcal{O}_d^T)} + \|y^0\|^2_{L^2(\Omega)}\right), \tag{112}$$

where $C = C(\|a\|_\infty, \tau_0, \tau_1, d_0, T) > 0$. Using (92) and the inequality (112), we obtain

$$\|\hat{\rho}_\varepsilon\|_{L^2(Q)} \leq C\left(\sum_{i=1}^{2}\left\|\frac{1}{\theta}z_{i,d}\right\|^2_{L^2(\mathcal{O}_d^T)} + \|y^0\|^2_{L^2(\Omega)}\right), \tag{113a}$$

$$\|\hat{\Psi}_{i\varepsilon}\|_{L^2((0,T);H_0^1(\Omega))} \leq C\left(\sum_{i=1}^{2}\left\|\frac{1}{\theta}z_{i,d}\right\|^2_{L^2(\mathcal{O}_d^T)} + \|y^0\|^2_{L^2(\Omega)}\right), \tag{113b}$$

where $C = C(\|a\|_\infty, \tau_0, \tau_1, d_0, T) > 0$. In view of (113a), we can extract subsequences still denoted by $\hat{\rho}_\varepsilon$ and $\hat{\Psi}_{i\varepsilon}$ such that when $\varepsilon \to 0$, we obtain

$$\hat{\rho}_\varepsilon \rightharpoonup \hat{\rho} \quad \text{weakly in } L^2(Q), \tag{114a}$$

$$\hat{\Psi}_{i\varepsilon} \rightharpoonup \hat{\Psi}_i \quad \text{weakly in } L^2((0,T);H_0^1(\Omega)). \tag{114b}$$

From (90), (105a) and (114a), we have (12). Proceeding as for convergence of $\hat{y}_\varepsilon$ in pages 23 and 24 while passing to the limit in (92), we prove using the convergence (114b) that $\hat{\Psi}_i$, $i = 1, 2$ satisfies (14). Passing to the limit in (91) while using (114a), we prove that $\hat{\rho}$ satisfies (13).

It then follows from (106), (108a), (109a) and (110) that $\hat{h}, \hat{y}, \hat{p}_1$ and $\hat{p}_2$ solve the null controllability problem (86)–(87) and (7). Finally, using the weak-lower semi-continuity of the norm and (105a), we deduce from (100) the estimate (15).

3. The semi-linear case

Now to prove the hierarchic control of the semi-linear system (1) is equivalent to prove that Theorem 1.2 holds true. We thus need to solve Problem 1.1 and Problem 1.2. To this end, we rewrite (1) as follows

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + f(y)\mathbf{1}_{\Omega\setminus\omega} = v_1\mathbf{1}_{\mathcal{O}_1} + v_2\mathbf{1}_{\mathcal{O}_2} + h\mathbf{1}_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0,.) = y^0 & \text{in } \Omega. \end{cases} \tag{115}$$

Then h is a control of system (115) if and only if $k = f(y) + h$ is a control of system (1). By writing (1) in the form (115), we bring the nonlinearity in a bounded sub-domain of Ω according to hypothesis (3). This fact is important to obtain the compactness properties required to apply the fixed point argument for dealing the nonlinear case.

Now, we will obtain an optimality system that characterizes any Nash quasi-equilibrium.

3.1. Characterization of Nash quasi-equilibrium

Here, we solve Problem 1.1 associated to (115). For any $h \in L^2(\omega^T)$, as the system (115) is nonlinear, the cost functions J_1 and J_2 are not convex in general. We consider a weaker concept of equilibrium and now, we look for the Nash quasi-equilibrium $\hat{v}_1 = \hat{v}_1(h) \in L^2(\mathcal{O}_1^T)$ and $\hat{v}_2 = \hat{v}_2(h) \in L^2(\mathcal{O}_2^T)$.

According to Definition 1.1, a pair $(\hat{v}_1, \hat{v}_2)$ is a Nash quasi-equilibrium of (115) and (4) associated to $h \in L^2(\omega^T)$ if (8) and (9) hold. This means that

$$\alpha_i \int_{\mathcal{O}_d^T} (\hat{y} - z_{i,d})Z_i \,\mathrm{d}x\,\mathrm{d}t + N_i \int_{\mathcal{O}_i^T} \hat{v}_i v_i \,\mathrm{d}x\,\mathrm{d}t = 0, \quad \text{for every } v_i \in L^2(\mathcal{O}_i^T), \tag{116}$$

where Z_i is solution of

$$\begin{cases} \dfrac{\partial Z_i}{\partial t} - \Delta Z_i + f'(\hat{y})\mathbf{1}_{\Omega\backslash\omega} Z_i = v_i \mathbf{1}_{\mathcal{O}_i} & \text{in } Q, \\ Z_i = 0 & \text{on } \Sigma, \\ Z_i(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{117}$$

In order to interpret (116), we consider the adjoint state $\hat{p}_i \in L^2((0,T); H_0^1(\Omega)) \cap \mathcal{C}([0,T]; L^2(\Omega))$, $i = 1, 2$ solution of

$$\begin{cases} -\dfrac{\partial \hat{p}_i}{\partial t} - \Delta \hat{p}_i + f'(\hat{y})\mathbf{1}_{\Omega\backslash\omega} \hat{p}_i = \alpha_i(\hat{y} - z_{i,d})\mathbf{1}_{\mathcal{O}_d} & \text{in } Q, \\ \hat{p}_i = 0 & \text{on } \Sigma, \\ \hat{p}_i(T,.) = 0 & \text{in } \Omega. \end{cases}$$

If we multiply the first equation in (117) by $\hat{p}_i$ and integrate by parts over Q, we obtain that

$$\alpha_i \int_{\mathcal{O}_d^T} Z_i(\hat{y} - z_{i,d})\, \mathrm{d}x\, \mathrm{d}t = \int_{\mathcal{O}_i^T} v_i \hat{p}_i\, \mathrm{d}x\, \mathrm{d}t = 0, \quad \text{for every } v_i \in L^2(\mathcal{O}_i^T),$$

which combining with (116) gives

$$\hat{v}_i = -\frac{1}{N_i}\hat{p}_i \quad \text{in } \mathcal{O}_i^T, \quad i = 1, 2.$$

We thus have proved the following results:

Proposition 3.1: *Let $h \in L^2(\omega^T)$. Then, the pair $(\hat{v}_1, \hat{v}_2)$ is a Nash quasi-equilibrium for functionals J_i, $i = 1, 2$ given by (4) if*

$$\hat{v}_i = -\frac{1}{N_i}\hat{p}_i \quad \text{in } \mathcal{O}_i^T, \quad i = 1, 2, \tag{118}$$

where $(\hat{y}, \hat{p}_i)$ is solution of the following systems

$$\begin{cases} \dfrac{\partial \hat{y}}{\partial t} - \Delta \hat{y} + \mathbf{1}_{\Omega\backslash\omega} f(\hat{y}) = h\mathbf{1}_\omega - \dfrac{1}{N_1}\hat{p}_1\mathbf{1}_{\mathcal{O}_1} - \dfrac{1}{N_2}\hat{p}_2\mathbf{1}_{\mathcal{O}_2} & \text{in } Q, \\ \hat{y} = 0 & \text{on } \Sigma, \\ \hat{y}(0,.) = y^0 & \text{in } \Omega \end{cases} \tag{119}$$

and

$$\begin{cases} -\dfrac{\partial \hat{p}_i}{\partial t} - \Delta \hat{p}_i + f'(\hat{y})\mathbf{1}_{\Omega\backslash\omega} \hat{p}_i = \alpha_i(\hat{y} - z_{i,d})\mathbf{1}_{\mathcal{O}_d} & \text{in } Q, \\ \hat{p}_i = 0 & \text{on } \Sigma, \\ \hat{p}_i(T,.) = 0 & \text{in } \Omega. \end{cases} \tag{120}$$

3.2. Proof of Theorem 1.2

To complete the proof of Theorem 1.2, we will solve Problem 1.2 associated to (119)–(120). We are interested in proving that, there exists a control $\hat{h} \in L^2(\omega^T)$ such that if $(\hat{y}, \hat{p}_i)$ is solution of (119)–(120), then (7) is satisfies.

We observe that, for any $\hat{y} \in L^2((0,T);L^2(\Omega \setminus \omega))$, we have

$$f(\hat{y}) - f(0) = a(\hat{y})\hat{y},$$

where $a(\hat{y})$ is defined by

$$a(\hat{y}) = \int_0^1 f'(\sigma \hat{y})\, \mathrm{d}\sigma. \tag{121}$$

Now, for any $z \in L^2((0,T);L^2(\Omega \setminus \omega))$, we consider the linearized system

$$\begin{cases} \dfrac{\partial \hat{y}}{\partial t} - \Delta \hat{y} + a(z)\mathbf{1}_{\Omega\setminus\omega}\hat{y} = h\mathbf{1}_{\omega} - \dfrac{1}{N_1}\hat{p}_1\mathbf{1}_{\mathcal{O}_1} - \dfrac{1}{N_2}\hat{p}_2\mathbf{1}_{\mathcal{O}_2} & \text{in } Q, \\ \hat{y} = 0 & \text{on } \Sigma, \\ \hat{y}(0,.) = y^0 & \text{in } \Omega, \end{cases} \tag{122}$$

where $\hat{p}_i,\ i = 1, 2$ is solution of

$$\begin{cases} -\dfrac{\partial \hat{p}_i}{\partial t} - \Delta \hat{p}_i + c(z)\mathbf{1}_{\Omega\setminus\omega}\hat{p}_i = \alpha_i(\hat{y} - z_{i,d})\mathbf{1}_{\mathcal{O}_d} & \text{in } Q, \\ \hat{p}_i = 0 & \text{on } \Sigma, \\ \hat{p}_i(T,.) = 0 & \text{in } \Omega, \end{cases} \tag{123}$$

with

$$c(z) = f'(z), \quad \forall\, z \in L^2((0,T);L^2(\Omega \setminus \omega)).$$

Since $f \in W^{1,\infty}(\mathbb{R})$, we have that

$$a \quad \text{and} \quad c \text{ belong to } L^\infty((0,T) \times (\Omega \setminus \omega)). \tag{124}$$

Now, we want to prove that systems (122)–(123) is null controllable. For this, we consider their following adjoint systems:

$$\begin{cases} -\dfrac{\partial \rho}{\partial t} - \Delta \rho + a(z)\mathbf{1}_{\Omega\setminus\omega}\rho = \alpha_1\Psi_1\mathbf{1}_{\mathcal{O}_{1,d}} + \alpha_2\Psi_2\mathbf{1}_{\mathcal{O}_{2,d}} & \text{in } Q, \\ \rho = 0 & \text{on } \Sigma, \\ \rho(T,\cdot) = \rho^T & \text{in } \Omega, \end{cases} \tag{125}$$

and for $i = 1, 2$,

$$\begin{cases} \dfrac{\partial \Psi_i}{\partial t} - \Delta \Psi_i + c(z)\mathbf{1}_{\Omega\setminus\omega}\Psi_i = -\dfrac{1}{N_i}\rho\mathbf{1}_{\mathcal{O}_i} & \text{in } Q, \\ \Psi_i = 0 & \text{on } \Sigma, \\ \Psi_i(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{126}$$

Proceeding as in Section 2.2, we show that the observability inequality associated with systems (125)–(126) is given by

$$\|\rho(0,\cdot)\|^2_{L^2(\Omega)} + \int_Q \frac{1}{\varpi^2}|\rho|^2\ \mathrm{d}x\,\mathrm{d}t + \sum_{i=1}^{2}\int_Q \theta^2|\Psi_i|^2\ \mathrm{d}x\,\mathrm{d}t \le C\int_{\omega^T} |\rho|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{127}$$

where $C = C(\|a\|_\infty, \|c\|_\infty, \tau_0, \tau_1, d_0, T) > 0$, the weight functions ϖ and θ are given by (81) and (84), respectively.

Now, using this observability inequality and proceeding as in the Section 2.3, we show that, for any $z \in L^2((0,T);L^2(\Omega \setminus \omega))$, there exists a control $\hat{h} = \hat{h}(z) \in L^2(\omega^T)$ such that if $(\hat{y}, \hat{p}_i)$ is solution of (122)–(123) then (7) is satisfied. Moreover, $\hat{h}$ satisfies (15).

We now consider a nonlinear map

$$S : L^2((0,T);L^2(\Omega \setminus \omega)) \to L^2((0,T);L^2(\Omega \setminus \omega))$$

such that, for every $z \in L^2((0,T);L^2(\Omega \setminus \omega))$, $S(z) = \hat{y}$ where $(\hat{y}, \hat{p}_i)$ are solutions of (122)–(123). Proving that S has a fixed point $\hat{y} \in L^2((0,T);L^2(\Omega \setminus \omega))$ will allows us to say that $\hat{y}$ is solution of (115) and consequently, will be sufficient to finish the proof of Theorem 1.2. To this end, we use the Schauder fixed-point theorem.

Proposition 3.2: (1) *S is continuous,*
(2) *S is compact,*
(3) *The range of S is bounded; i.e.*

$$\exists M > 0 : \ \|S(z)\|_{L^2((0,T);L^2(\Omega\setminus\omega))} \leq M, \quad \forall\, z \in L^2((0,T);L^2(\Omega \setminus \omega)).$$

Proof: Throughout the rest of this work, the expression $\|\cdot\|_{\Omega\setminus\omega}$ will denote $\|\cdot\|_{L^\infty((0,T)\times(\Omega\setminus\omega))}$.

(1) *S* is continuous.

Let (z_n) be a sequence such that $z_n \to z$ strongly in $L^2((0,T);L^2(\Omega \setminus \omega))$. Then we can extract a subsequence of (z_n) denoted (z_{n_k}) such that

$$z_{n_k} \to z \quad \text{almost everywhere (a.e.) in } (0,T) \times (\Omega \setminus \omega).$$

Therefore, f being a function of class $\mathcal{C}^1$, the functions a and c are continuous and we have

$$\begin{aligned} a(z_{n_k}) &\to a(z) \quad \text{a.e. in } (0,T) \times (\Omega \setminus \omega), \\ c(z_{n_k}) &\to c(z) \quad \text{a.e. in } (0,T) \times (\Omega \setminus \omega). \end{aligned}$$

It then follows from (124) and the Lebesgue dominated convergence theorem that

$$a(z_{n_k}) \to a(z) \quad \text{strongly in } L^2((0,T) \times (\Omega \setminus \omega)), \tag{128a}$$

$$c(z_{n_k}) \to c(z) \quad \text{strongly in } L^2((0,T) \times (\Omega \setminus \omega)). \tag{128b}$$

As Theorem 1.1 holds for every $z \in L^2((0,T);L^2(\Omega \setminus \omega))$, it also holds for $z_{n_k} \in L^2((0,T);L^2(\Omega \setminus \omega))$. Thus the control $\hat{h}_{n_k} = \hat{h}(z_{n_k})$ is such that $\hat{y}_{n_k} = \hat{y}(z_{n_k})$ satisfies

$$\left\{ \begin{array}{ll} \dfrac{\partial \hat{y}_{n_k}}{\partial t} - \Delta \hat{y}_{n_k} + a(z_{n_k})\mathbf{1}_{\Omega\setminus\omega}\hat{y}_{n_k} = \tau_{n_k} & \text{in } Q, \\ \hat{y}_{n_k} = 0 & \text{on } \Sigma, \\ \hat{y}_{n_k}(0,.) = y^0 & \text{in } \Omega, \end{array} \right. \tag{129}$$

where $\tau_{n_k} = \hat{h}_{n_k}\mathbf{1}_\omega - \frac{1}{N_1}\hat{p}_{1,n_k}\mathbf{1}_{\mathcal{O}_1} - \frac{1}{N_2}\hat{p}_{2,n_k}\mathbf{1}_{\mathcal{O}_2}$ and $\hat{p}_{i,n_k}$, $i = 1,2$ is solution of

$$\left\{ \begin{array}{ll} -\dfrac{\partial \hat{p}_{i,n_k}}{\partial t} - \Delta \hat{p}_{i,n_k} + c(z_{n_k})\mathbf{1}_{\Omega\setminus\omega}\hat{p}_{i,n_k} = \alpha_i(\hat{y} - z_{i,d})\mathbf{1}_{\mathcal{O}_d} & \text{in } Q, \\ \hat{p}_{i,n_k} = 0 & \text{on } \Sigma, \\ \hat{p}_{i,n_k}(T,.) = 0 & \text{in } \Omega, \end{array} \right. \tag{130}$$

$$\hat{y}_{n_k}(T,.) = 0 \quad \text{in } \Omega \tag{131}$$

and

$$\hat{v}_{i,n_k} = -\frac{\hat{p}_{i,n_k}}{N_i} \quad \text{in } \mathcal{O}_i^T, \quad i = 1, 2. \tag{132}$$

Moreover,

$$\hat{h}_{n_k} = \hat{\rho}_{n_k} \quad \text{in } \omega^T, \tag{133}$$

with $\hat{\rho}_{n_k}$ solution of

$$\begin{cases} -\dfrac{\partial \hat{\rho}_{n_k}}{\partial t} - \Delta \hat{\rho}_{n_k} + a(z_{n_k})\mathbf{1}_{\Omega\setminus\omega}\hat{\rho}_{n_k} = \displaystyle\sum_{i=1}^{2} \alpha_i \hat{\Psi}_{i,n_k}\mathbf{1}_{\mathcal{O}_d} & \text{in } Q, \\ \hat{\rho}_{n_k} = 0 & \text{on } \Sigma, \\ \hat{\rho}_{n_k}(T,\cdot) = \rho^T & \text{in } \Omega, \end{cases} \tag{134}$$

and $\hat{\Psi}_{i,n_k}$, $i = 1, 2$, solution of

$$\begin{cases} \dfrac{\partial \hat{\Psi}_{i,n_k}}{\partial t} - \Delta \hat{\Psi}_{i,n_k} + c(z_{n_k})\mathbf{1}_{\Omega\setminus\omega}\hat{\Psi}_{i,n_k} = -\dfrac{1}{N_i}\hat{\rho}_{n_k}\mathbf{1}_{\mathcal{O}_i} & \text{in } Q, \\ \hat{\Psi}_{i,n_k} = 0 & \text{on } \Sigma, \\ \hat{\Psi}_{i,n_k}(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{135}$$

In addition, $\hat{h}_{n_k}$ and $\hat{v}_{i,n_k}$, $i = 1, 2$ verify (15) and (107), respectively. This means that there exists a positive constant $C_1 = C(\|a\|_{\Omega\setminus\omega}, \|c\|_{\Omega\setminus\omega}, T, \tau_0, \tau_1, d_0) > 0$ and $C_2 = C(\|a\|_{\Omega\setminus\omega}, \|c\|_{\Omega\setminus\omega}, T, \alpha_1, \alpha_2, N_1, N_2, \tau_0, \tau_1, d_0) > 0$ such that

$$\|\hat{h}_{n_k}\|_{L^2(\omega^T)} \leq C_1 \left(\sum_{i=1}^{2} \left\| \frac{1}{\theta} z_{i,d} \right\|^2_{L^2(\mathcal{O}_d^T)} + \|y^0\|^2_{L^2(\Omega)} \right)^{1/2} \tag{136}$$

and for $i = 1, 2$,

$$\|\hat{v}_{i,n_k}\|_{L^2(\mathcal{O}_i^T)} \leq C_2 \left(\|y^0\|_{L^2(\Omega)} + \sum_{i=1}^{2} \|z_{i,d}\|_{L^2(\mathcal{O}_d^T)} + \sum_{i=1}^{2} \left\| \frac{1}{\theta} z_{i,d} \right\|_{L^2(\mathcal{O}_d^T)} \right). \tag{137}$$

We set

$$W(Q) = \left\{ \rho \in L^2((0,T); H_0^1(\Omega)), \ \frac{\partial \rho}{\partial t} \in L^2((0,T); H^{-1}(\Omega)) \right\}. \tag{138}$$

Then observing that $(\hat{y}_{n_k})$ and $(\hat{p}_{in_k})$, $i = 1, 2$ are solutions of (129) and (130), respectively, using (124), (136) and (137), we prove that there exists $C = C(\|a\|_{\Omega\setminus\omega}, \|c\|_{\Omega\setminus\omega}, T, \alpha_1, \alpha_2, N_1, N_2, \tau_0, \tau_1, d_0) > 0$ such that

$$\|\hat{y}_{n_k}\|_{W(Q)} \leq C \left(\|y^0\|_{L^2(\Omega)} + \sum_{i=1}^{2} \|z_{i,d}\|_{L^2(\mathcal{O}_d^T)} + \sum_{i=1}^{2} \left\| \frac{1}{\theta} z_{i,d} \right\|_{L^2(\mathcal{O}_d^T)} \right), \tag{139a}$$

$$\|\hat{p}_{i,n_k}\|_{W(Q)} \leq C \left(\|y^0\|_{L^2(\Omega)} + \sum_{i=1}^{2} \|z_{i,d}\|_{L^2(\mathcal{O}_d^T)} + \sum_{i=1}^{2} \left\| \frac{1}{\theta} z_{i,d} \right\|_{L^2(\mathcal{O}_d^T)} \right). \tag{139b}$$

Thus, there exist $\hat{h} \in L^2(\omega^T)$, $\hat{v}_i \in L^2(\mathcal{O}_i^T)$, and $\hat{y}$, $\hat{p}_i$, $i = 1, 2$ in $W(Q)$ such that

$$\hat{h}_{n_k} \rightharpoonup \hat{h} \quad \text{weakly in } L^2(\omega^T), \tag{140a}$$

$$\hat{v}_{i,n_k} \rightharpoonup \hat{v}_i \quad \text{weakly in } L^2(\mathcal{O}_i^T), \quad i=1,2, \tag{140b}$$

$$\hat{y}_{n_k} \rightharpoonup \hat{y} \quad \text{weakly in } W(Q), \tag{140c}$$

$$\hat{p}_{i,n_k} \rightharpoonup \hat{p}_i \quad \text{weakly in } W(Q), \quad i=1,2. \tag{140d}$$

It follows from Aubin-Lions Lemma that

$$\hat{y}_{n_k} \to \hat{y} \quad \text{strongly in } L^2((0,T)\times(\Omega\setminus\omega)), \tag{141a}$$

$$\hat{p}_{i,n_k} \rightharpoonup \hat{p}_i \quad \text{strongly in } L^2((0,T)\times(\Omega\setminus\omega)), \quad i=1,2. \tag{141b}$$

Therefore, proceeding as for the convergence of $(\hat{y}_\varepsilon, \hat{p}_{i,\varepsilon})$ in Pages 22–24, we prove by passing to the limit in systems (129)–(130) while using (140a)–(140d), (128a) and (141a)–(141b) that, $(\hat{h}, \hat{y}, \hat{p}_1, \hat{p}_2)$ satisfies the null controllability problem (86)–(87) and (7).

Applying (68) to $(\hat{\rho}_{n_k})$ and $(\hat{\Psi}_{in_k})$, $i=1,2$ and using (111), we deduce that

$$\int_Q |\hat{\rho}_{n_k}|^2 \,\mathrm{d}x\,\mathrm{d}t + \sum_{i=1}^{2}\int_Q |\hat{\Psi}_{i,\varepsilon}|^2 \,\mathrm{d}x\,\mathrm{d}t \le \int_{\omega^T} |\hat{\rho}_{n_k}|^2 \,\mathrm{d}x\,\mathrm{d}t$$

$$\le C\left(\sum_{i=1}^{2}\left\|\frac{1}{\theta}z_{i,d}\right\|^2_{L^2(\mathcal{O}_d^T)} + \|y^0\|^2_{L^2(\Omega)}\right),$$

where $C = C(\|a\|_{\Omega\setminus\omega}, \|c\|_{\Omega\setminus\omega}, T, \tau_0, \tau_1, d_0) > 0$. This implies that

$$\begin{aligned} \hat{\rho}_{n_k} &\rightharpoonup \hat{\rho} \quad \text{weakly in } L^2(Q), \\ \hat{\Psi}_{n_k} &\rightharpoonup \hat{\Psi}_i \quad \text{weakly in } L^2((0,T);H_0^1(\Omega)). \end{aligned} \tag{142}$$

Therefore proceeding as for the convergence of $\hat{\Psi}_{i\varepsilon}$, $i=1,2$ and $\hat{\rho}_\varepsilon$ pages 24–25 while passing to the limit in (134) and (135), we prove using (128a) and (142) that $\hat{\rho}$ and $\hat{\Psi}_i$, $i=1,2$ satisfy (13) and (14), respectively. Moreover, $\hat{h}$ satisfies (12).

(2) S is compact.

The operator is compact. To prove that we proceed exactly as in [25], using on the one hand the fact that $\hat{y} \in W((0,T)\times(\Omega\setminus\omega))$ and on the other hand the compact embedding of $W((0,T)\times(\Omega\setminus\omega))$ into $L^2((0,T);L^2(\Omega\setminus\omega))$.

(3) The range of S is bounded.

Let $z \in L^2((0,T);L^2(\Omega\setminus\omega))$. Since $S(z) = \hat{y}(z)$ is solution of (122) with $\hat{h}$ satisfying (15) and $\hat{v}_i$, $i=1,2$ verify (106) and (107), we prove that there exists

$C = C(\|a\|_{\Omega\setminus\omega}, \|c\|_{\Omega\setminus\omega}, T, \alpha_1, \alpha_2, N_1, N_2,$
$\tau_0, \tau_1, d_0) > 0$ such that

$$\|\hat{y}\|_{L^2((0,T);L^2(\Omega\setminus\omega))} \le C\left(\|y^0\|_{L^2(\Omega)} + \sum_{i=1}^{2}\|z_{i,d}\|_{L^2(\mathcal{O}_d^T)} + \sum_{i=1}^{2}\left\|\frac{1}{\theta}z_{i,d}\right\|_{L^2(\mathcal{O}_d^T)}\right).$$

It then follows from Proposition 3.2 that the operator S has a fixed point $\hat{y}$. Since $\hat{k} = f(\hat{y}) + \hat{h}$, then the proof of Theorem 1.2 is complete. ■

3.3. Equilibria and quasi-equilibria

In this section, we will prove Proposition 1.1. As we said at the beginning of the Section 3, solving system (1) with control $\hat{k} = f(\hat{y}) + \hat{h}$ is equivalent to solving system (115) with control $\hat{h}$. By writing (1) in the form (115) is very important because, we bring the nonlinearity in a bounded sub-domain of Ω according to hypothesis (3). So, in this section, we will consider the system (115).

Let us suppose that the nonlinearity f satisfies (2) and $f \in W^{2,\infty}(\mathbb{R})$, let $\hat{h} \in L^2(\omega^T)$ be given and let $(\hat{v}_1, \hat{v}_2)$ be the associated Nash quasi-equilibria. Let also $w_1, w_2 \in L^2(\mathcal{O}_1^T)$. Our aim is to estimate the second derivative $D_1^2 J_1(h; \hat{v}_1, \hat{v}_2) \cdot (w_1, w_1)$.

For any $s \in \mathbb{R}$, let us denote by $\hat{y}^s$ the solution of the following system

$$\begin{cases} \dfrac{\partial \hat{y}^s}{\partial t} - \Delta \hat{y}^s + f(\hat{y}^s)\mathbf{1}_{\Omega\backslash\omega} = (\hat{v}_1 + s w_1)\mathbf{1}_{\mathcal{O}_1} + \hat{v}_2 \mathbf{1}_{\mathcal{O}_2} + \hat{h}\mathbf{1}_\omega & \text{in } Q, \\ \hat{y}^s = 0 & \text{on } \Sigma, \\ \hat{y}^s(0,.) = y^0 & \text{in } \Omega \end{cases} \tag{143}$$

and let us set $\hat{y} := \hat{y}^s|_{s=0}$.

Now, we have

$$\begin{aligned} & D_1 J_1(\hat{h}; \hat{v}_1 + s w_1, \hat{v}_2) \cdot w_2 - D_1 J_1(\hat{h}; \hat{v}_1, \hat{v}_2) \cdot w_2 \\ & \quad = s N_1 \int_{\mathcal{O}_1^T} w_1 w_2 \,\mathrm{d}x\,\mathrm{d}t + \alpha_1 \int_{\mathcal{O}_d^T} (\hat{y}^s - z_{1,d}) z^s \,\mathrm{d}x\,\mathrm{d}t - \alpha_1 \int_{\mathcal{O}_d^T} (\hat{y} - z_{1,d}) z \,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{144}$$

where z^s is the derivative of the state $\hat{y}^s$ with respect to $\hat{v}_1 + s w_1$ in the direction w_2, i.e. the solution to

$$\begin{cases} \dfrac{\partial z^s}{\partial t} - \Delta z^s + f'(\hat{y}^s)\mathbf{1}_{\Omega\backslash\omega} z^s = w_1 \mathbf{1}_{\mathcal{O}_1} & \text{in } Q, \\ z^s = 0 & \text{on } \Sigma, \\ z^s(0,.) = 0 & \text{in } \Omega. \end{cases} \tag{145}$$

We will also use the notation $z := z^s|_{s=0}$.

Let us introduce the adjoint of (145)

$$\begin{cases} -\dfrac{\partial \hat{p}^s}{\partial t} - \Delta \hat{p}^s + f'(\hat{y}^s)\mathbf{1}_{\Omega\backslash\omega} \hat{p}^s = \alpha_1 (\hat{y}^s - z_{1,d})\mathbf{1}_{\mathcal{O}_d} & \text{in } Q, \\ \hat{p}^s = 0 & \text{on } \Sigma, \\ \hat{p}^s(T,.) = 0 & \text{in } \Omega \end{cases} \tag{146}$$

and let us use the notation $\hat{p} := \hat{p}^s|_{s=0}$.

Mutiplying the first equation of (145) by $\hat{p}^s$ solution of (146) and integrating by part over Q, we obtain

$$\alpha_1 \int_Q (\hat{y}^s - z_{1,d})\mathbf{1}_{\mathcal{O}_d} z^s \,\mathrm{d}x\,\mathrm{d}t = \int_Q w_2 \hat{p}^s \mathbf{1}_{\mathcal{O}_1} \,\mathrm{d}x\,\mathrm{d}t. \tag{147}$$

From (144) and (147), we have

$$\begin{aligned} D_1 J_1(h; \hat{v}_1 + s w_1, \hat{v}_2) \cdot w_2 - D_1 J_1(h; \hat{v}_1, \hat{v}_2) \cdot w_2 = s N_1 & \int_{\mathcal{O}_1^T} w_1 w_2 \,\mathrm{d}x\,\mathrm{d}t \\ & + \int_{\mathcal{O}_1^T} (\hat{p}^s - \hat{p}) w_2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{148}$$

Notice that

$$-\frac{\partial}{\partial t}(\hat{p}^s-\hat{p})-\Delta(\hat{p}^s-\hat{p})+\left[f'(\hat{y}^s)-f'(\hat{y})\right]\mathbf{1}_{\Omega\setminus\omega}\hat{p}^s+f'(\hat{y})\mathbf{1}_{\Omega\setminus\omega}(\hat{p}^s-\hat{p})=\alpha_1(\hat{y}^s-\hat{y})\mathbf{1}_{\mathcal{O}_d}$$

and

$$\frac{\partial}{\partial t}(\hat{y}^s-\hat{y})-\Delta(\hat{y}^s-\hat{y})+\left[f'(\hat{y}^s)-f'(\hat{y})\right]\mathbf{1}_{\Omega\setminus\omega}=sw_1\mathbf{1}_{\mathcal{O}_1}.$$

Consequently, the following limits

$$\eta=\lim_{s\to 0}\frac{1}{s}(\hat{p}^s-\hat{p})\quad\text{and}\quad \phi=\lim_{s\to 0}\frac{1}{s}(\hat{y}^s-\hat{y})$$

exist and satisfy

$$\begin{cases} -\dfrac{\partial\eta}{\partial t}-\Delta\eta+f'(\hat{y})\mathbf{1}_{\Omega\setminus\omega}\eta+f''(\hat{y})\mathbf{1}_{\Omega\setminus\omega}\phi\hat{p}=\alpha_1\phi\mathbf{1}_{\mathcal{O}_d} & \text{in } Q,\\ \eta=0 & \text{on } \Sigma,\\ \eta(T,.)=0 & \text{in } \Omega \end{cases}\tag{149}$$

and

$$\begin{cases} \dfrac{\partial\phi}{\partial t}-\Delta\phi+f'(\hat{y})\mathbf{1}_{\Omega\setminus\omega}\phi=w_1\mathbf{1}_{\mathcal{O}_1} & \text{in } Q,\\ \phi=0 & \text{on } \Sigma,\\ \phi(0,.)=0 & \text{in } \Omega. \end{cases}\tag{150}$$

Thus, from (148) and (149)–(150), we deduce that

$$D_1^2J_1(\hat{h};\hat{v}_1,\hat{v}_2)\cdot(w_1,w_2)=\int_{\mathcal{O}_1^T}\eta w_2\,\mathrm{d}x\,\mathrm{d}t+N_1\int_{\mathcal{O}_1^T}w_1w_2\,\mathrm{d}x\,\mathrm{d}t.\tag{151}$$

In particular, for $w_2=w_1$, we have

$$D_1^2J_1(\hat{h};\hat{v}_1,\hat{v}_2)\cdot(w_1,w_1)=\int_{\mathcal{O}_1^T}\eta w_1\,\mathrm{d}x\,\mathrm{d}t+N_1\int_{\mathcal{O}_1^T}|w_1|^2\,\mathrm{d}x\,\mathrm{d}t.\tag{152}$$

Let us show that, for some constant $C_1>0$ independent of $\hat{h},\ \eta,\ \phi,\ w_1$, one has

$$\left|\int_{\mathcal{O}_1^T}\eta w_1\,\mathrm{d}x\,\mathrm{d}t\right|\le C_1(1+\|\hat{h}\|_{L^2(\omega^T)})\|w_1\|_{L^2(\mathcal{O}_1^T)}.\tag{153}$$

Indeed, given $w_1\in L^2(\mathcal{O}_1^T)$ and since $f'\in L^\infty((0,T)\times(\Omega\setminus\omega))$, from the energy estimate, we have

$$\|\phi\|^2_{L^2(Q)}+\|\nabla\phi\|^2_{L^2(Q)}\le C\|w_1\|^2_{L^2(\mathcal{O}_1^T)}.\tag{154}$$

Using systems (149) and (150), we have

$$\begin{aligned}\int_{\mathcal{O}_1^T}\eta w_1\,\mathrm{d}x\,\mathrm{d}t&=\int_Q\eta\left(\frac{\partial\phi}{\partial t}-\Delta\phi+f'(\hat{y})\mathbf{1}_{\Omega\setminus\omega}\phi\right)\mathrm{d}x\,\mathrm{d}t\\ &=\int_Q\phi\left(-\frac{\partial\eta}{\partial t}-\Delta\eta+f'(\hat{y})\mathbf{1}_{\Omega\setminus\omega}\eta\right)\mathrm{d}x\,\mathrm{d}t\\ &=\int_Q\phi\left(-f''(\hat{y})\mathbf{1}_{\Omega\setminus\omega}\phi\hat{p}+\alpha_1\phi\mathbf{1}_{\mathcal{O}_d}\right)\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$= \int_Q -f''(\hat{y})\mathbf{1}_{\Omega\setminus\omega}|\phi|^2\hat{p}\,\mathrm{d}x\,\mathrm{d}t + \int_Q \alpha_1|\phi|^2\mathbf{1}_{\mathcal{O}_d}\,\mathrm{d}x\,\mathrm{d}t. \tag{155}$$

Applying Hölder inequality in the above expression and using the fact that $f'' \in L^\infty((0,T)\times(\Omega\setminus\omega))$, we have

$$\left|\int_{\mathcal{O}_1^T} \eta w_1\,\mathrm{d}x\,\mathrm{d}t\right| \leq \|f''\|_{\Omega\setminus\omega}\|\phi\|^2_{L^{2r'}((0,T);L^{2s'}(\Omega\setminus\omega))}\|\hat{p}\|_{L^r((0,T);L^s(\Omega\setminus\omega))} + \alpha_1\|\phi\|^2_{L^2(\mathcal{O}_d^T)}, \tag{156}$$

where r' and s' are the conjugate of r and s, respectively. To bound the right-hand side of this latter inequality, the idea is to find r and s such that

$$\hat{p} \in L^r((0,T);L^s(\Omega\setminus\omega)), \quad \phi \in L^{2r'}((0,T);L^{2s'}(\Omega\setminus\omega)).$$

First, we have that $\phi \in L^2((0,T);H^2(\Omega\setminus\omega)) \cap L^\infty((0,T);H^1(\Omega\setminus\omega))$. It is reasonable to look for which values of d and b the following embedding holds:

$$L^2((0,T);H^2(\Omega\setminus\omega)) \cap L^\infty((0,T);H^1(\Omega\setminus\omega)) \hookrightarrow L^d((0,T);L^b(\Omega\setminus\omega)). \tag{157}$$

Using interpolation results, we deduce that

$$\frac{1}{d} = \frac{\theta}{2}, \quad 0 < \theta < 1. \tag{158}$$

From Sobolev embedding results, we have

$$H^2(\Omega\setminus\omega) \hookrightarrow L^{\frac{2n}{n-4}}(\Omega\setminus\omega), \tag{159a}$$

$$H^1(\Omega\setminus\omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega\setminus\omega). \tag{159b}$$

Then, the space $L^b(\Omega\setminus\omega)$ is an intermediate space with respect to (159a) and (159b) if

$$\frac{1}{b} = \frac{(n-4)\theta}{2n} + \frac{(n-2)(1-\theta)}{2n}, \quad 0 < \theta < 1. \tag{160}$$

Taking $d = 2r'$ and $b = 2s'$, it follows that appropriate values of r and s are

$$r = \frac{d}{d-2} \quad \text{and} \quad s = \frac{\mathrm{d}n}{2(d+2)}.$$

On the other hand, since $\hat{h} \in L^2(\omega^T)$, $\hat{v}_i \in L^2(\mathcal{O}_i^T)$ and $y^0 \in L^2(\Omega)$, we have that

$$\hat{y} \in L^2((0,T);H^1(\Omega\setminus\omega)) \cap L^\infty((0,T);L^2(\Omega\setminus\omega)) \hookrightarrow L^{\bar{\mathrm{d}}}((0,T);L^{\bar{b}}(\Omega\setminus\omega)). \tag{161}$$

Using the interpolation argument, we obtain that

$$\bar{b} = \frac{2\bar{\mathrm{d}}n}{\bar{\mathrm{d}}n-4}.$$

From parabolic regularity, we have

$$\phi \in L^{\bar{\mathrm{d}}}((0,T);W^{2,\bar{b}}(\Omega\setminus\omega)) \hookrightarrow L^{\bar{\mathrm{d}}}\left((0,T);L^{\frac{n\bar{b}}{n-2\bar{b}}}(\Omega\setminus\omega)\right) = L^{\bar{\mathrm{d}}}\left((0,T);L^{\frac{2\bar{\mathrm{d}}n}{\bar{\mathrm{d}}(n-4)-4}}(\Omega\setminus\omega)\right). \tag{162}$$

Taking, $\bar{\mathrm{d}} = r$, it follows that $\phi \in L^{\bar{\mathrm{d}}}((0,T);L^{\frac{2\,\mathrm{d}n}{d(n-8)+8}}(\Omega\setminus\omega))$ and, in order to have $\phi \in L^r((0,T);L^s(\Omega\setminus\omega))$, we need

$$\frac{\mathrm{d}n}{2(\mathrm{d}+2)} \leq \frac{2\mathrm{d}n}{\mathrm{d}(n-8)+8},$$

which is true if and only if $n \leq 12$.

Now, using (154) and the standard energy estimate of systems (143) for $s = 0$ and (146) for $s = 0$, the term (155) becomes

$$\left| \int_{\mathcal{O}_1^T} \eta w_1 \, dx \, dt \right| \leq C_1(1 + \|\hat{h}\|_{L^2(\omega^T)}), \tag{163}$$

where C_1 is a positive constant independent of N_1 and N_2.

Combining (152) and (163), it follows that

$$D_1^2 J_1(\hat{h}; \hat{v}_1, \hat{v}_2) \cdot (w_1, w_1) \geq \left[N_1 - C_1(1 + \|\hat{h}\|_{L^2(\omega^T)}) \right] \|w_1\|^2_{L^2(\mathcal{O}_1^T)}, \quad \forall\, w_1 \in L^2(\mathcal{O}_1^T).$$

In a similar way, we can prove that there exists a positive constant C_2 independent of N_1 and N_2 such that

$$D_2^2 J_2(\hat{h}; \hat{v}_1, \hat{v}_2) \cdot (w_2, w_2) \geq \left[N_2 - C_2(1 + \|\hat{h}\|_{L^2(\omega^T)}) \right] \|w_2\|^2_{L^2(\mathcal{O}_1^T)}, \quad \forall\, w_2 \in L^2(\mathcal{O}_2^T).$$

Now, taking N_i such that $N_i > C_i(1 + \|\hat{h}\|_{L^2(\omega^T)})$, then the functionals J_i, $i = 1, 2$ given by (4) are convex and therefore the pair $(\hat{v}_1, \hat{v}_2)$ is a Nash equilibrium in the sense of (5)–(6). Since $\hat{k} = f(\hat{y}) + \hat{h}$, then the proof of Proposition 1.1 is complete.

4. Conclusion

In this paper, we proved that system (1) is Stackelberg–Nash null controllable in an unbounded domain. The results have been obtained under the following assumptions: $\mathcal{O}_i \cap \omega = \emptyset$, the set $(\Omega \setminus \omega)$ is bounded and $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$.

Let us mention that, in the linear case, the quadratic functionals are convex and then we look for a Nash equilibrium. But, in the semi-linear case, we don't have the convexity of those functionals in general. That is the reason why we redefine the concept of equilibrium and then, we now look for a Nash-quasi equilibrium. Next, we show that under certain conditions, there is a equivalence between Nash equilibrium and quasi-Nash equilibrium.

Acknowledgements

I would like to express my gratitude to the unknown referees for helpful advice.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

The author was supported by the German Academic Exchange Service (D.A.A.D) under the Scholarship Program PhD AIMS-Cameroon.

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